THE JOURNAL OF SYMBOLIC LOGIC
Volume 34, Number 2, June 1969

FINITE INSEPARABILITY OF SOME THEORIES OF CYLINDRIFICATION ALGEBRAS

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An elementary theory T in a language L is (*strongly*) *finitely inseparable* if the set of logically valid sentences of L and the set of T-finitely refutable sentences are recursively inseparable. In §1 we establish a sufficient condition for the elementary theory of a class of BA's with operators to be finitely inseparable. This is done using the methods developed independently by M. Rabin and D. Scott (see [6]) on the one hand and by Ershov on the other (see [2]). Our condition applies, in particular (see Corollary 5), to the class of all cylindric (polyadic) algebras of dimension α $(1 < \alpha < \omega)$. As a consequence of finite inseparability we obtain the undecidability of the elementary theories of each of the classes of algebras listed in Corollary 5. For various classes of cylindric algebras the undecidability was first established by Tarski (cf. [4]) who obtained in certain cases the stronger conclusion that the identities holding in the classes are recursively unsolvable. An interesting feature of the proof presented here is that it does not use diagonal elements, which play a central role in Tarski's proof. In §2 we make some observations concerning the decision problem for cylindrification algebras of dimension 1.

The author wishes to thank Professor Donald Monk for his valuable suggestions concerning the preparation of this paper. In particular, the formulation of the Rabin-Scott method presented below was partially adopted from a course given by Professor Monk at the University of Colorado.

§0. We assume as known the basic notions of logic. For concepts related to decision procedures the reader may consult the survey article [2]. By a language L we mean a first-order predicate calculus with equality containing only a finite number of nonlogical constants. A theory T in L is a logically closed set of sentences of L. If T is a theory in L, we denote by T_{fin} the theory of all finite models of T; a sentence of L is T-(finitely) refutable if it fails in some (finite) model of T. For a formula ϕ of L with free variables included among $v_0, \cdots, v_{n-1}$ and a structure $\mathfrak{A}$ of the appropriate type, we denote by $\phi^{\mathfrak{A}}$ the n-ary relation $\{x \in {}^nA : x$ satisfies ϕ in $\mathfrak{A}\}$ induced on $\mathfrak{A}$.

We now describe the Rabin-Scott procedure for establishing finite inseparability; a version of this procedure formulated in terms of undecidability and for a language with just one binary predicate is given in Rabin [6].

Let L_0 and L_1 be languages, where we assume the only nonlogical constants of L_0 are predicate symbols. A *translation* of L_0 into L_1 is a pair (θ, f) where θ is a

Received November 5, 1967.

[1] Research supported by a predoctoral NSF Traineeship.

formula of L_1 with only one free variable v_0 and f is a function mapping the set of predicate symbols of L_0 into the set of formulas of L_1 such that if P is an m-ary predicate symbol of L_0, $f(P)$ is a formula of L_1 with free variables among $v_0, \cdots, v_{m-1}$. The function f extends to a function f^+ from the set of formulas of L_0 into the formulas of L_1 in the natural way—the quantifier $\forall v_i$ is handled by relativizing to $\theta(v_i)$ (cf. [8, p. 24]). Since L_0, L_1 contain only a finite number of nonlogical constants, f^+ is effective.

If $\mathfrak{B}$ is an L_1 structure and $\theta^{\mathfrak{B}}$ is nonempty we define $\mathfrak{B}_{L_0}$ as the L_0 structure $(\theta^{\mathfrak{B}}, R_P)_{P \in S}$ where S is the set of nonlogical constants of L_0 and for each m-ary predicate symbol P, R_P is the restriction of $f(P)^{\mathfrak{B}}$ to ${}^m(\theta^{\mathfrak{B}})$. The following fact about translations (θ, f) of L_0 into L_1 is basic: suppose $\mathfrak{B}$ is an L_1 structure, ψ a formula of L_0 and $x \in {}^\omega(\theta^{\mathfrak{B}})$, then

(1) x satisfies ψ in $\mathfrak{B}_{L_0}$ iff x satisfies $f^+(\psi)$ in $\mathfrak{B}$.

Let L^c be the language obtained from the language L by adjoining a new individual constant c. For a structure $\mathfrak{A}$ of L and $x \in A$, we denote by $\mathfrak{A}(x)$ the structure of L^c obtained from $\mathfrak{A}$ by interpreting c as the element x.

The following theorem which is due to Rabin and Scott is essentially a reformulation of Theorems 1 and 2 of [6] (cf. Theorem 3.3.2 of [2]) into the context of finite inseparability.

THEOREM 1. *Suppose T is a finitely inseparable theory in a language L all of whose nonlogical constants are predicate symbols and K is a class of L_0 structures. Further, suppose (θ, f) is a translation of L into $L_0{}^c$ and the following condition holds:*

(M_0) *for every finite model $\mathfrak{A}$ of T there is a finite $\mathfrak{B} \in K$ and $b \in B$ such that $\mathfrak{B}(b)_L \cong \mathfrak{A}$.*

Then Th (K) *is finitely inseparable.*

PROOF. We define a recursive function g from the sentences of L into the sentences of L_0 as follows: for a sentence ϕ of L let $g(\phi) = \forall v_m(\exists v_0(\theta(v_0))' \rightarrow (f^+(\phi))')$ where a primed formula is obtained from the unprimed one by substituting the least variable, v_m, not occurring in $\exists v_0 \theta(v_0) \rightarrow f^+(\phi)$ for the individual variable c in the unprimed formula. Using (1) we can easily verify that the function g has the following two properties:

(a) if ϕ is a logically valid sentence of L, then $g(\phi)$ is a logically valid sentence of L_0;

(b) if $\phi \notin T_{\text{fin}}$ then $g(\phi) \notin \text{Th}(K)_{\text{fin}}$.

The theorem follows from (a) and (b).

When a theory T and a class of L_0 structures K, related as above, satisfy the condition (M_0) we will say that the pair (T, K) satisfies (M_0).

Suppose K_0 and K_1 are classes of algebras of similarity types σ and τ respectively. We say that K_1 is *equationally definitionally embeddable* (e.d.e.) in K_0 if, for each σ_i-ary operation in the language of K_0, there is a σ_i-ary polynomial t_i in the language of K_1 such that if $\mathfrak{A} = (A, g_i)_{i \in Dmn\tau} \in K_1$, then the algebra $\mathfrak{A}^* = (A, t_i{}^{\mathfrak{A}})_{i \in Dmn\sigma} \in K_0$ where $t_i{}^{\mathfrak{A}}$ is the σ_i-ary operation induced on $\mathfrak{A}$ by t_i. Let $K_1{}^* = \{A^*: \mathfrak{A} \in K_1\}$. For a formula ϕ let ϕ^τ be the formula obtained from ϕ by replacing every σ_i-ary operation symbol g_i in the language of K_0 occurring in ϕ by the corresponding term t_i.

The following lemma is quite useful.

LEMMA 2. *Suppose K_0 and K_1 are classes of algebras of similarity types σ and τ and formulated in languages L_0 and L_1 respectively. Assume that K_1 is e.d.e. in K_0 and (θ, f) is a translation of L into the expanded language $L_0{}^c$. If we define $\theta' = \theta^\tau$ and the function f' on a predicate symbol P of L by $f'(P) = f(P)^\tau$, then*

(i) *(θ', f') is a translation of L into $L_1{}^c$;*

(ii) *if T is a theory in L for which $(T, K_1{}^*)$ satisfies (M_0), then (T, K_1) also satisfies (M_0);*

(iii) *if T is a finitely inseparable theory in L for which $(T, K_1{}^*)$ satisfies (M_0), then* Th K_1 *is also finitely inseparable.*

Part (i) of the lemma is obvious; part (ii) holds since $\mathfrak{A}(x)_L{}^* = \mathfrak{A}(x)_L$ for $\mathfrak{A} \in K_1$ and $x \in A$. This fact is easily proved by a standard induction argument on formulas. Part (iii) is immediate from (ii).

By the theory of two disjoint equivalence relations we mean the theory which has two binary relation symbols R and S as its only nonlogical constants and the following axioms:

$$\forall v_0 \forall v_1 (v_0 = v_1 \leftrightarrow v_0 R v_1 \wedge v_0 S v_1)$$
$$\forall v_0 \forall v_1 \forall v_2 (v_1 R v_0 \wedge v_2 R v_0 \rightarrow v_2 R v_1)$$
$$\forall v_0 \forall v_1 \forall v_2 (v_0 S v_1 \wedge v_0 S v_2 \rightarrow v_2 S v_1).$$

The following unpublished result of M. O. Rabin and D. Scott will be useful in the following section.

THEOREM 3.[2] *The theory T of two disjoint equivalence relations is finitely inseparable.*

PROOF. Let T_0 be the theory of an irreflexive, symmetric, binary relation Q; by an appropriate modification of the proof of Theorem 6 of [6] (cf. Theorem 3.3.3 of [2]) it follows that T_0 is finitely inseparable. Let $\theta(v_0)$ be the formula $\forall v_1(v_1 S v_0 \leftrightarrow v_1 = v_0)$ and $f(Q)$ the formula $\exists v_2 \exists v_3 (v_0 R v_2 \wedge v_2 S v_3 \wedge v_3 R v_1)$. Then (θ, f) is a translation of the language L of T_0 into the language of T. For Theorem 3 to follow from Theorem 1 with K equal the class of models of T it suffices to show that (T_0, K) satisfies (M_0). Let $\mathfrak{A} = (A, Q)$ be a finite model of T_0; let $B = Q \cup \{(x, x): x \in A\}$ and define $\mathfrak{B} = (B, R, S)$ where, for all $(x, y), (u, v) \in B$,

$$(x, y) R (u, v) \text{ iff } x = u,$$
$$(x, y) S (u, v) \text{ iff } (x, y) = (u, v) \text{ or } (x = v \text{ and } y = u).$$

Now $\mathfrak{B}$ is a finite model of T and $\mathfrak{A} \underset{h}{\cong} \mathfrak{B}_L$ where the desired isomorphism h is defined for all $x \in A$ by $hx = \{(x, x)\}$.

§1. For an ordinal number α a *cylindrification algebra of dimension* α is a structure $\mathfrak{A} = (A, +, \cdot, -, 0, 1, c_\kappa)_{\kappa < \alpha}$ such that $0, 1 \in A$, $-$, c_κ are unary operations on A, $+, \cdot$ are binary operations on A, and the following conditions hold in $\mathfrak{A}$ for all $\kappa, \lambda < \alpha$ and $x, y \in A$;

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[2] The proof of Theorem 3 given here was presented by Professor Scott in a course at the University of California in 1963 and is included with his permission.

(c_0) the structure $(A, +, \cdot, -, 0, 1)$ is a BA;
(c_1) $c_\kappa 0 = 0$;
(c_2) $x \leq c_\kappa x$;
(c_3) $c_\kappa(x \cdot c_\kappa y) = c_\kappa x \cdot c_\kappa y$;
(c_4) $c_\kappa c_\lambda x = c_\lambda c_\kappa x$.

We denote the class of all cylindrification algebras of dimension α by Cy_α. For an ordinal α and a nonempty set X, let $\mathfrak{A}(\alpha, X)$ denote the Cy_α

$$(S({}^\alpha X), \cup, \cap, \sim, 0, {}^\alpha X, C_\kappa)_{\kappa<\alpha}$$

where, for $Y \in S({}^\alpha X)$, i.e., $Y \subseteq {}^\alpha X$, and $\kappa < \alpha$ we define

$$C_\kappa Y = \{f \in {}^\alpha X\colon \text{for some } g \in Y, f_i = g_i \text{ for all } i \in \alpha \sim \{\kappa\}\}.$$

As an extension of the terminology of [4] we call subalgebras of $\mathfrak{A}(\alpha, X)$ *cylindrification set algebras* (a CyS_α) and a subdirect product of CyS_α's a *representable cylindrification algebra* (a RCy_α).

We consider the following condition on a class K of BA's with operators which is e.d.e. in Cy_α.

(H_0) For every $i < \omega$ there is an algebra $\mathfrak{A}_i \in K$ such that $\mathfrak{A}_i^* = \mathfrak{A}(\alpha, X_i)$ for some nonempty set X_i and for all $i < j < \omega$, $|X_i| < |X_j| < \omega$.

We now prove the main theorem of this note.

THEOREM 4. *For* $1 < \alpha < \omega$ *if* K *is a class of BA's with operators e.d.e. in* Cy_α *such that* (H_0) *holds in* K, *then* Th K *is finitely inseparable.*

PROOF. Suppose α is fixed, $1 < \alpha < \omega$, and K satisfies our hypothesis. Let $L_0(L_\alpha)$ be the language of $K(Cy_\alpha)$ and $L_0{}^c(L_\alpha{}^c)$ obtained from $L_0(L_\alpha)$ by adding a new individual constant c. Let T be the theory of two disjoint equivalence relations R and S. By Theorem 3 T is finitely inseparable. In view of Theorem 1 and Lemma 2 it suffices to show there is a translation (θ, f) of R, S into $L_\alpha{}^c$ such that (T, K^*) satisfies (M_0). The translation (θ, f) is given by:

$$\theta\colon v_0 \text{ is an atom and } v_0 \leq c,$$
$$fR\colon \theta(v_0) \wedge \theta(v_1) \wedge (c_0 v_0 = c_0 v_1),$$
$$fS\colon \theta(v_0) \wedge \theta(v_1) \wedge (c_1 v_0 = c_1 v_1).$$

Suppose $\mathfrak{A} = (A, R, S)$ is a finite model of T, i.e., R and S are equivalence relations on A such that for $x, y \in A$, if xRy and xSy, then $x = y$. Suppose the R-(equivalence) classes are R_i for $i < n$ and the S-classes are S_i for $i < m$. Set $\mathfrak{B} = \mathfrak{A}(\alpha, X_r)$ where r is the least s such that $\max\{m, n\} \leq |X_s|$ and $\mathfrak{A}(\alpha, X_s) \in K^*$. Choose a one-one function h from $\max\{m, n\}$ into X_r and define the function k from A into ${}^\alpha X_r$ by letting $kx = (h_i, h_j, h_0, \cdots, h_0)$ where i and j are the unique natural numbers such that $x \in S_i \cap R_j$. Define $gx = \{kx\}$ for $x \in A$ and let b be the range of k. It is easy to verify that g is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}(b)_L$, L the language of T. Thus (M_0) holds and the proof of Theorem 4 is complete.

We now give the more important classes K to which the previous theorem applies. The notation is from [4].

COROLLARY 5. Th K *is finitely inseparable for each of the following classes* K:

(i) Cy_α *and* RCy_α *for* $1 < \alpha < \omega$;
(ii) CA_α *and* RCA_α *for* $1 < \alpha < \omega$;

(iii) *(representable) polyadic algebras of degree* α *for* $1 < \alpha < \omega$;
(iv) *(representable) polyadic algebras of degree* α *with equality for* $1 < \alpha < \omega$;
(v) *(representable) relation algebras*;
(vi) *projective algebras.*

That the classes of algebras in (v), (vi) satisfy the hypothesis of Theorem 4 is proved in [5, p. 135, p. 938]. As an immediate consequence of Corollary 5 we obtain

COROLLARY 6. *If* K *is one of the classes listed above, then* Th K *and* (Th $K)_{\mathrm{fin}}$ *are undecidable.*

If K is one of the classes in 5(ii), 5(iv), or 5(v), the undecidability of Th K is due to A. Tarski.

If we relax our notion of language in order to allow us to speak about algebras with countably many operations, a version of Theorem 4 still holds. Call an elementary theory T in L *R-inseparable* if the sets of logically valid and T-refutable sentences of L are recursively inseparable. Theorem 1 may be modified to obtain the conclusion that Th K is R-inseparable if condition (M_0) is replaced by the condition (M_1) obtained from (M_0) by removing the stipulation that $\mathfrak{B}$ is necessarily finite. Clearly, Lemma 2 carries over into our more general context where, in conclusion (ii), (M_0) is replaced by (M_1). The proof of Theorem 4, for $\alpha = \omega$, can now be repeated verbatim to prove that Th K is R-inseparable. Consider the following condition on a class K of BA's with (at most countably many) operators which is e.d.e. in Cy_α:

(H_1) There is an algebra $\mathfrak{A} \in K$ such that $\mathfrak{A}^* = \mathfrak{A}(\alpha, X)$ where X is infinite.

The above and an obvious modification of the proof of Theorem 4 yields:

THEOREM 7. *For* $1 < \alpha \leq \omega$ *if* K *is a class of BA's with (at most countably many) operators e.d.e. in* Cy_α *such that either* (H_0) *or* (H_1) *holds in* K, *then* Th K *is R-inseparable.*

COROLLARY 8. *If* K_α *is one of the classes listed in* 5(i) *to* 5(iv) *then* Th (K_ω) *is R-inseparable and hence undecidable.*

Again, in the case of CA_ω's (RCA_ω's) the undecidability was first proved by Tarski (see [4]). In view of the fact that the finite CA_ω's are just the discrete CA_ω's (essentially BA's), (Th $CA_\omega)_{\mathrm{fin}}$ is decidable; hence finite inseparability fails in case $\alpha = \omega$.

§2. In this section we draw some conclusions related to the decision problem for Cy_1's (essentially the class of CA_1's) using standard methods. For a class K of Cy_1's we denote by Simple K, AtK, $(P_f K)PK$, and K_{fin} the classes of simple algebras of K, atomic algebras of K, (finite) direct products of members of K, and finite algebras of K, respectively. We begin with a few observations.

(1) Th (Simple Cy_1) and Th (Simple $AtCy_1$) are decidable.

The formula $(x = 0 \wedge y = 0) \vee [\neg(x = 0) \wedge y = 1]$ serves as a possible definition of $c_0 x = y$; hence (1) follows from the decidability of the theory of BA's (cf. p. 20 of [8] and [7]). We may clearly extend Tarski's characterization [7] of the elementary types of atomic BA's to a description of the elementary types of simple atomic Cy_1's.

(2) $$\text{Th (Simple } Cy_{1\,\text{fin}}) = \text{Th (Simple } AtCy_1).$$

The nontrivial part of (2) holds since any simple atomic infinite Cy_1 is elementarily equivalent to an ultraproduct of simple finite Cy_1's. As a consequence of (1), (2) and the result of Feferman and Vaught [3] that if a class K of algebras has a decidable theory, then so does the class $P_f K$, we obtain the following theorem.

THEOREM. Th $(Cy_{1\,\text{fin}}) = (\text{Th } Cy_1)_{\text{fin}}$ *is decidable.*

We conclude by showing that Th $(Cy_{1\,\text{fin}})$ is the same as the theory of all complete atomic Cy_1's (i.e., Th (Complete $AtCy_1$)) and is hence decidable. In view of the inclusions Th $(Cy_{1\,\text{fin}}) \supseteq$ Th (Complete $AtCy_1$) $\supseteq$ Th (P(Simple $AtCy_1$)) it suffices to show that Th $(Cy_{1\,\text{fin}})$ = Th (P(Simple $AtCy_1$)). By Theorems 6.7 and 6.8 of [3], Th (P(Simple $AtCy_1$)) = Th (P_f(Simple $AtCy_1$)); hence we only need to show that Th (P_f(Simple $AtCy_1$)) = Th (P_f(Simple $Cy_{1\,\text{fin}}$)) (=Th $(Cy_{1\,\text{fin}})$). The nontrivial part of this equality is proven similar to (2) once we observe that two finite products $\mathfrak{A}$ and $\mathfrak{B}$ of Simple $AtCy_1$'s are elementarily equivalent iff for each elementary type K of Simple $AtCy_1$'s $\mathfrak{A}$ and $\mathfrak{B}$ have the same number of simple factors in K. Thus we conclude that Th $Cy_{1\,\text{fin}}$ = Th (Complete $AtCy_1$).

Actually there is a larger class of Cy_1's which has a decidable theory, namely, the class of all reduced products of Simple Cy_1. (This is immediate from Ershov's result in [1] that if a class K of algebras has a decidable theory, then so does the class of all reduced products.) The author has obtained examples to be published later showing that this theory is different from Th Cy_1 and, in fact, showing that the class of Cy_1's elementarily equivalent to reduced products of simple Cy_1's is indeed very meager. This contrasts with Ershov's result [1] that every BA is elementarily equivalent to a reduced power of the 2-element BA.

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