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THE STANDARD SEMIGROUP OF OPERATORS OF A VARIETY

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For a class K of similar algebras let $H(K)$, $S(K)$, and $P(K)$ denote, respectively, the class of all homomorphic images, subalgebras, and isomorphs of products of algebras in K. To the set of symbols $\{H, S, P\}$ we can associate a partially ordered semigroup S in a natural way. S is a quotient of the free semigroup generated by $\{H, S, P\}$; two words σ and τ are identified if $\sigma(K)=\tau(K)$ for every class K of similar algebras ($\sigma(K)$ is K if σ is the empty word) and $\sigma \leqslant \tau$ in S if $\sigma(K) \subseteq \tau(K)$ for every class K of similar algebras. Of course, by considering symbols associated with other algebraic operations we can obtain partially ordered semigroups other than S.

E. Nelson proved in [1] that the semigroups obtained from $\{H, S, P, P_s\}$, $\{C, H, S, P, P_r\}$, and $\{C, H, S, P_r, P_u\}$ are finite. (P_s, P_r, P_u, C denote the subdirect product, reduced product, ultraproduct, and covering operations respectively.) C. Platt has announced that the semigroup obtained from the direct limit operation and the one from the inverse limit operation are infinite. For S, which is finite by Nelson's result, even more is known.

D. Pigozzi announced a complete description of S in [3]. Its elements are the words: 0(empty word), H, S, P, HS, SH, HP, PH, SP, PS, PSH, SPH, SHP, PHS, HPS, $SPHS$, $SHPS$, and HSP. Further, the ordering among these 18 words has the following diagram (see page 17).

In addition to the structure of the compositions of operators in general, their structure relativized to the closed classes determined by the operators can also be considered. For a variety V let S(V) be the partially ordered semigroup obtained as above for $\{H, S, P\}$ except that K is restricted to subclasses of V. S(V) is always a homomorphic image of S. In this terminology the following is a consequence of what Pigozzi proved. Let V_τ denote the variety of all algebras having similarity type τ.

THEOREM. (1) *If V is the variety of all commutative semigroups,* S(V)=S. (2) *If τ contains at least one operation with rank at least* 2, *then* S(V_τ)=S.

The following shows that the restriction on the similarity type in (2) is necessary.

THEOREM. *If τ consist only of unary operations, then* S(V_τ) *has* 13 *elements. The nontrivial relationships among words of* S *that hold in* S(V_τ) *are generated by* $HS=SH$,

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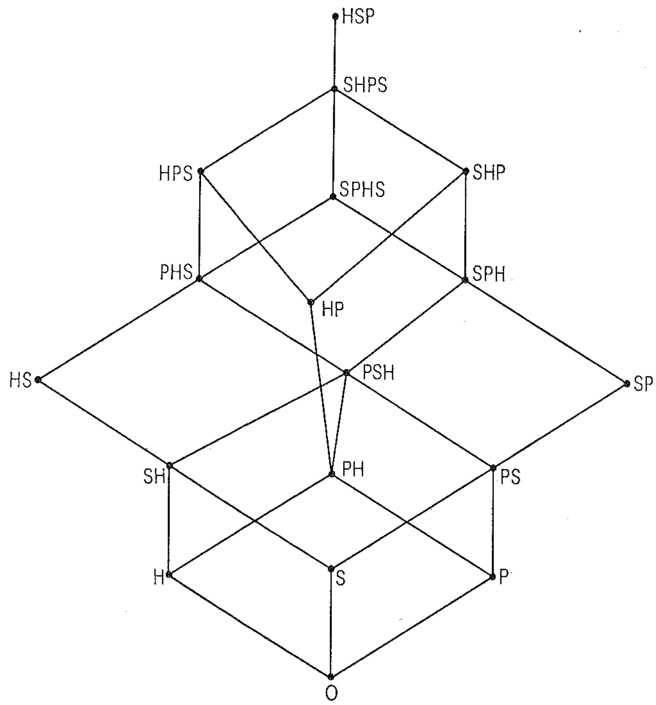

Diagram 1.

i.e., the congruence relation associated with the natural homomorphism from **S** *onto* $\mathbf{S}(V_\tau)$ *is generated by* $\{(HS, SH)\}$.

For various important varieties V the structure of $\mathbf{S}(V)$ has been either completely or nearly determined. These results are summarized in the following table. In the second column we give the cardinality of $\mathbf{S}(V)$ and in the third we list the generators of the nontrivial relationships among the words of **S** which hold in $\mathbf{S}(V)$.

V	$\lvert S(V)\rvert$	nontrivial relationships
Boolean algebras	11	$HS=SH$, $SP=HSP$
Distributive lattices	11	$HS=SH$, $SP=HSP$
Lattices	17 or 18	?
Abelian groups	13	$HS=SH$
Commutative semigroups	18	None
Groups	18	None

In each of the above results the main part of the proof is the construction of examples showing that elements of a proposed $\mathbf{S}(V)$ are distinct. For lattices it is not known if there is a class K of lattices for which $SHPS(K)$ is not a variety. If such a class exist $\mathbf{S}(\text{Lattices})=\mathbf{S}$; on the other hand, if $SHPS(K)$ is a variety for every class K of lattices, $SHPS=HSP$ holds for lattices and $\mathbf{S}(\text{Lattices})$ has 17 elements. The problem of finding a class K of groups for which $SHPS(K)$ is not a variety was solved by P. Neumann [2]. Consequently, $\mathbf{S}(\text{Groups})=\mathbf{S}$.

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