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ARITHMETIC PROPERTIES OF RELATIVELY FREE PRODUCTS

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Arithmetic properties of direct products have been studied for many years. W. Hanf showed in [3] that the cancellation law, Cantor-Bernstein and square-root properties fail for direct products of Boolean algebras. The present note contains some observations concerning analogous problems for free products. Free product is understood to mean coproduct where the canonical injections are monic. Unlike direct products, the free product of algebras depends on the variety where it is formed and it may not even exist. Free products are assumed to exist in any variety considered. Consider the following three properties for algebras A,B,C in a variety V.

(1) $A * B \cong A * C$ implies $B \cong C$.

(2) $A \cong B * D$ and $B \cong A * C$ implies $A \cong B$.

(3) $B * B \cong C * C$ implies $B \cong C$.

Properties (1),(2),(3) are known as the cancellation law, Cantor-Berstein property and square-root property, respectively. These properties are established in section 1 under suitable finiteness assumptions. Counterexamples to (1),(2),(3) are given for Boolean algebras in section 2. In section 3 the results from section 2 are applied to derive counterexamples for other classes of algebraic structures.

1. We first consider the cancellation property (1) for a variety V. Normally this property will obviously fail if we do not require some finiteness condition on A. For example, it usually fails if we let A be a V-free algebra generated by an infinite set and let B, C be V-free algebras generated by finite sets with different cardinalities. For a subclass K of V we say that A _cancels for_ K if (1) holds for all B, C in K. The results below give conditions under which A cancels for the class of all finite members of V.

Theorem 1.1. Suppose $A * B \cong A * C$ for A, B, C in V and, in addition, B, C are finite and $0 < |\mathrm{Hom}(A,X)| < \omega$ for every subalgebra X of B and every subalgebra X of C. Then $B \cong C$.

Proof. From the conditions on A, B, C, X and the fact that $|\mathrm{Hom}(A,X)| \cdot |\mathrm{Hom}(B,X)| = |\mathrm{Hom}(A*B,X)| = |\mathrm{Hom}(A,X)| \cdot |\mathrm{Hom}(C,X)|$ it follows that

(1.2) $|\mathrm{Hom}(B,X)| = |\mathrm{Hom}(C,X)|$ for every $X \in S\{B,C\}$.

Let $\{X_i : i < n\}$, for some n, be a listing without repetition of the maximal proper subalgebras of X. For an algebra D the principle of inclusion-exclusion gives

$$|\mathrm{Epi}(D,X)| = |\mathrm{Hom}(D,X)| - \sum_i |\mathrm{Hom}(D,X_i)| + \sum_{i,j} |\mathrm{Hom}(D,X_i \cap X_j)| - \cdots .$$

The right sides of the two equations obtained by letting $D = B$ and $D = C$ are the same by (1.2). Thus,

(1.3) $|\mathrm{Epi}(B,X)| = |\mathrm{Epi}(C,X)|$ for every $X \in S\{B,C\}$.

Setting $X = B$ in (1.3) gives $0 \neq \mathrm{Epi}(B,B) = \mathrm{Epi}(C,B)$ so $|C| \geq |B|$. Similarly, setting $X = C$ in (1.3) gives an epimorphism from B onto C. $|C| \geq |B|$ implies this map is an isomorphism so $B \cong C$.

The above proof was obtained by "dualizing" the proof of the analogous result for direct products due to L. Lovász [5]. The following statements are immediate corollaries of (1.1).

(1.4) If $A * B \cong A * C$ for A,B,C in V where B,C are finite, A finitely generated and $\text{Hom}(A,X) \neq 0$ for every $X \in S\{B,C\}$ then $B \cong C$.

(1.5) If every member of V has a one element subalgebra, then every finitely generated member cancels for the finite members of V.

(1.6) Finitely generated V-free algebras cancel for the finite members of V.

In particular, (1.5) applies for any variety of lattices or groups. By (1.4) finite Boolean algebras cancel for the class of finite BA's.

We now consider the Cantor-Bernstein property (2) and the square-root property (3) for the finite members of a variety. The message of (1.7) and (1.8) is that the finite versions hold. The proof of (1.7) depends on the following lemma due to Bjarni Jónsson.

Lemma. Suppose A,B in V and A is not isomorphic to a proper subalgebra of itself. Then $A \cong A * B$ if and only if, for every extension E of A and every homomorphism $h:B \to E$, h maps B into A.

Proof. Given E and h let $k = 1_A * h:A * B \to E$. Let i,j denote the canonical embeddings of A,B into $A * B$. By the assumption on A, the monic $A \overset{i}{\to} A * B \cong A$ is onto so $i(A) = A * B$. Hence $h(B) = k(j(B)) \leq k(i(A)) = A$ as desired. Conversely, consider $i:A \to A * B \leftarrow B:j$ and let $E = A * B$ (identifying $i(A)$ with A) and $h = j$. Then $j(B) \leq i(A)$ and so $A \cong A * B$.

Theorem 1.7. The Cantor-Bernstein property (2) holds in V whenever A is not isomorphic to a proper subalgebra of itself. In particular, it holds whenever A is finite.

Proof. (2) implies $A \cong A * C * D$; so $C * D$ satisfies the condition of the Lemma and there exist a map of D into A. Then C satisfies the condition of the lemma; so $A \cong A * C$ as desired.

Two proofs of the square-root property for finite algebras are given below. The first one uses the argument in 1.1. The second was communicated to me by Jan Mycielski who reported that it was discovered a few years ago by A. Ehrenfeucht (unpublished). Ehrenfeucht's proof is outlined below since it illustrates an alternative way of giving the counting argument basic to both 1.8 and 1.1.

Theorem 1.8. The square-root property (3) holds whenever B,C are finite members of V.

Proof. For every $X \in S\{B,C\}$,

$$|\mathrm{Hom}(B,X)|^2 = |\mathrm{Hom}(B * B,X)| = |\mathrm{Hom}(C * C,X)| = |\mathrm{Hom}(C,X)|^2;$$

hence, (1.2) holds. Thus, $B \cong C$ by the same argument used in the proof of 1.1.

The key to 1.8 and also 1.1 is to show that, for finite algebras B and C, (1.2) implies (1.3).

Ehrenfeucht proved (1.3) by induction on $|X|$. For $|X| = 1$ the result is trivial. Now, for any algebra A,

$$|\mathrm{Hom}(A,X)| = |\mathrm{Epi}(A,X)| + \sum_{D \subsetneqq X} |\mathrm{Epi}(A,D)|.$$

In the two equations obtained by letting $A = B$ and $A = C$, the left sides are equal by (1.2) and the right terms on the right side are equal by the induction hypothesis. Hence $|\mathrm{Epi}(B,X)| = |\mathrm{Epi}(C,X)|$ follows.

2. We now turn our attention to some counterexamples. To minimize our work we introduce property (4) below. Property (2) is clearly equivalent to the statement that $A \cong A * C * D$ implies $A \cong A * C$. This statement, in turn, implies

(4) $A \cong A * C * C$ implies $A \cong A * C$.

Observe that (3) also implies (4); for if there exist A and C where $A \ncong A * C$ but $A \cong A * C * C$, then $(A * C) * (A * C) \cong A * A$ while $A \ncong A * C$.

For BA's we will show that (1) fails with A finite and (4) fails with C finite (and thus, (2), (3) also fail). The examples in (2.4), (2.5) are based on those due to Hanf and Tarski in [3] for direct products. We need the following from [3].

(2.1) There exist denumerable BA's B,C such that $B^2 \cong C^2$ and $B \not\cong C$.

(2.2) For each integer $n > 1$ there exist a BA H_n such that $H_n \cong H_n \times 2^n$ but $H_n \not\cong H_n \times 2^k$ for $k = 1,\ldots,n-1$. The H_n's are uncountable and $H_n^2 \cong H_n$.

The following simple observation is crucial.

(2.3) If A,B are BA's and B is finite with n atoms then $A * B \cong A^n$ (the direct product of A with itself n times).

Proof. The dual space of $A * B$ is the cartesian product of the dual X of A times an n element discrete space. Thus, it is also a disjoint union of n copies of X.

Theorem 2.4 (1) The four element BA 2^2 does not cancel for the class of all denumerable BA's.

(2) The two element BA is the only finite BA to cancel for the

class of BA's.

Proof. (1) Let $A = 2^2$ and choose B, C from (2.1). By (2.3), $A * B \cong B^2 \cong C^2 \cong A * C$ but $B \not\cong C$.

(2). Suppose $A = 2^n$ for $n > 1$. Let $B = H_n$ and $C = H_n \times 2$. From (2.2), $B \not\cong C$. However, using (2.3) and (2.2), $A * C \cong (H_n \times 2)^n \cong (H_n \times 2)^n \cong H_n^n \times 2^n \cong H_n^n \cong A * B$.

Theorem 2.5. Property (4) fails for BA's with C finite.

Proof. Let $C = 2^2$ and $A = H_3 \times 2$. By (2.3) and (2.2), $A * C * C \cong (H_3 \times 2)^4 \cong H_3^4 \times 2^4 \cong (H_3^3 \times 2^3) \times 2 = A$ but $A * C \cong H_3 \times 2^2 \not\cong A$.

It is worth noting the counterexample to (3) obtained from (2.5) is $A = H_3 \times 2$ and $B = H_3 \times 2^2$. By passing to the dual spaces we get Boolean spaces X, Y which are not homeomorphic however, $X \times X$ is homeomorphic to $Y \times Y$. This answers an old question posed by Halmos in [2].

3. We can obtain counterexamples to the arithmetic properties (1), (2), and (3) for other classes of algebras from the results in the previous section by constructing appropriate functors.

Proposition 3.1. Suppose $\mathcal{C}$ is a category (with free products) and Γ is a full embedding of the category of BA's into $\mathcal{C}$ that preserves free products. Then (1), (2), (3), and (4) fail in $\mathcal{C}$. Moreover, $\Gamma(2^n)$ does not cancel for $\mathcal{C}$ for finite $n > 2$.

The proof is straightforward using the fact that a full embedding has the property: $A \cong B$ iff $\Gamma A = \Gamma B$.

We apply (3.1) below to obtain examples for the variety generated by a primal algebra, bounded distributive lattices, and rings. In each case the functor used is one that arises in the study of sectional representations over Boolean spaces. Let X_B denote the Stone space of a BA B. For a universal algebra A and a Boolean space X let $\Gamma(X,A)$ denote the algebra of all continuous functions from X into A (given the discrete topology). For each of the varieties (categories) $\mathcal{C}$ to be considered a natural algebra A is selected in $\mathcal{C}$. The functor Γ from BA's into $\mathcal{C}$ is defined for a BA B by $\Gamma(B) = \Gamma(X_B,A)$. Γ does the natural thing to homomorphisms. This functor Γ is always an embedding. To apply (3.1) we have to check in the categories below that Γ is full and preserves free products.

(3.2) $\mathcal{C}$ is the variety generated by a primal algebra A.

In this case the functor Γ establishes an equivalence between the category of BA's and $\mathcal{C}$ (see Hu[4]). Thus, (3.1) applies and its conclusion holds for $\mathcal{C}$. In particular, A^2 does not cancel for $\mathcal{C}$.

(3.3) $\mathcal{C}$ is the variety of (0,1)-distributive lattice.

Let A be the two element distributive lattice with 0 and 1 distinguished. The functor Γ in this situation essentially just forgets the complementations operation. The embedding is full since any 0,1 preserving lattice homomorphism between BA's also preserves complements. The following lemma (3.4) implies that Γ preserves free products.

(3.4) If B_i is a Boolean subalgebra of a BA B $(i \in I)$ and $B = \Pi^*_I B_i$ (as BA's), then $B = \Pi^*_I B_i$ as (0,1)-distributive lattices.

The proof of (3.4) easily follows from the internal description of free products of BA's and a similar description for bounded distributive lattices due to Grätzer and Lakser [1].

Thus, (3.1) applies and its conclusion holds for the class of bounded distributive lattices. In particular, 2^n does not cancel for this class for $1 < n < \omega$.

(3.5) Rings.

Let A denote a fixed field and $\mathcal{C}$ the category of all commutative rings that contain A as a subring. Mappings in $\mathcal{C}$ are A-homomorphisms. The free product operation in $\mathcal{C}$ is the tensor product $\otimes_A$ over A (see Zariski, Samuel [6]). The ring $\Gamma(B) = \Gamma(X_B, A)$ is in $\mathcal{C}$ when A is identified with the subring of constant functions on X_B. Observe that $\Gamma(B)$ is commutative and B is isomorphic to the BA $B(\Gamma(B))$ of all idempotent elements of $\Gamma(B)$. In fact, $B(\Gamma(B))$ is just the double dual space of B. It easily follows that the embedding Γ is full. We need the following lemma.

(3.6) For Boolean spaces $X, Y, \Gamma(X,A) \otimes_A \Gamma(Y,A) \cong (X \times Y, A)$.

The obvious projections of $X \times Y$ onto X and Y induce embeddings $f: \Gamma(X,A) \to \Gamma(X \times Y, A)$ and $g: \Gamma(Y,A) \to \Gamma(X \times Y, A)$. Let $R = f(\Gamma(X,A))$ and $S = g(\Gamma(Y,A))$. Note that $\sigma \in \Gamma(X \times Y, A)$ is in R if and only $\{\sigma^{-1}(a) : a \in A\}$ partitions $X \times Y$ into disjoint sets of the form $N \times Y$ where N is a clopen subset of X. A similar description of S also holds.

For N, N' clopen subsets of X,Y respectively, let $c_{N \times N'}$ denote the characteristic function of $N \times N'$. For $a \in A$ $ac_{N \times N'} = (ac_{N \times Y}) \cdot (c_{X \times N'})$ is in the subring generated by R and S; thus, R and S generate $\Gamma(X \times Y, A)$.

Below we need the observation that $(f(\sigma') \cdot g(\tau'))(x,y) = \sigma'(x) \cdot \tau'(y)$ for all $\sigma' \in \Gamma(X,A)$, $\tau' \in \Gamma(Y,A)$, $x \in X$, $y \in Y$.

To verify (3.6) it remains to show R and S are linearly disjoint over A(cf., Zariski-Samuel [6]). Suppose $\sigma_1,\ldots,\sigma_n \in R$ and $\tau_1,\ldots,\tau_m \in S$ are each linearly independent sets over A. For each i,j choose $\sigma_i' \in \Gamma(X,A)$ and $\tau_j' \in \Gamma(Y,A)$ such that $f(\sigma_i') = \sigma_i$ and $g(\tau_j') = \tau_j$. In addition consider $c_{ij} \in A$ such that $\sum_{i,j} c_{ij}\sigma_i\tau_j$.

For $(x,y) \in X \times Y$, the observation above shows that

$$0 = \sum_{i,j} c_{ij}(\sigma_i\tau_j)(x,y) = \sum_i(\sum_j c_{ij}\tau_j'(y))\sigma_i'(x).$$

Since $\sigma_1',\ldots,\sigma_n'$ are linearly independent in $\Gamma(X,A)$ over A, it follows that $0 = \sum_j c_{ij}\tau_j'(y)$ for each j and $y \in Y$. The linear independence of $\tau_1',\ldots,\tau_m'$ in $\Gamma(Y,A)$ over A now implies that $c_{ij} = 0$ for each i,j. Thus, $\{\sigma_i\tau_j : i = 1,\ldots,n;\ j = 1,\ldots,m\}$ is linearly independent over A; so R and S are linearly disjoint and consequently (3.6) holds.

From (3.6) it follows that Γ preserves free products and hence (3.1) applies. So (1), (2), (3), and (4) fail in $\mathcal{C}$. In particular, A^n $(1 < n < \omega)$ does not cancel for the category of all commutative rings that contain the field A.

References

1. Grätzer, H. Lakser, Chain conditions in distributive free products of lattices. Trans. Amer. Math. Soc. 144 (1969), 301-302.

2. P.R. Halmos, Lectures on Boolean Algebras. Mathematical Studies No. 1, 1963. Van Nostrand, Princeton, N.J.

3. W. Hanf, On Some Fundamental Problems Concerning Isomorphism of Boolean Algebras. Math. Scand. 5 (1957), 205-217.

4. T.K. Hu, Stone Duality for Primal Algebra Theory. Math Z. 110 (1969), 180-198.

5. L. Lovász, On the Cancellation Law among Finite Relational Structures, Periodica Math. Hung. 1 (1971), 145-156.

6. O. Zariski, P. Samuel, Commutative Algebra, I. University Series in Higher Mathematics, 1958. Van Nostrand, Princeton, N.J.