Restricted Direct Products and Sectional Representations*

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Abstract. A construction of restricted direct product decompositions is given. It is based on the general algebraic theory of sectional representations associated with BOOLEan algebras of factor relations. Restricted direct product decompositions of an algebra A are shown to correspond in a one-one way with certain BOOLEan structures of factor relations on A.

The technique of obtaining a sectional representation of an algebra from a BOOLEan algebra of factor relations has become standard (see Bibliography of [4]). The results in this paper are motivated by the belief that this framework is suitable for constructing and comparing different types of product representations of an algebra. We test this belief on the product notions due to HASHIMOTO, the restricted direct products. We use this notion because it is general enough to include both weak and strong direct products as well as various intermediate product notions. One of our objectives is to show that every restricted direct product can be obtained (up to equivalence) from a sectional representation associated with a suitable BOOLEan algebra. Another goal is the characterization of the BOOLEan algebra structures that correspond to restricted direct products.

In section 1 we review the basic construction of sectional representations. In section 2 we discuss restricted product decompositions, their equivalence, and associate a BOOLEan algebra $\mathscr{B}_h$ of factor relations on A with a restricted product decomposition h of A. The class of $\mathscr{B}_h'$ s is characterized in section 4 while in section 3, we show that h can be recovered from $\mathscr{B}_h$. The net effect of this is that an equivalence class of equivalent restricted product decompositions of A corresponds to a certain BOOLEan structure and this correspondence is one-one. In section 5 the correspondence is applied to direct product decompositions.

1. Sectional representations

The construction of a sectional representation of a universal algebra given in COMER [1] is outlined below.

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Suppose A is an algebra and $\mathscr{B} = (B, +, \cdot, 0, 1)$ is a BOOLEAN algebra (BA). We call $\mathscr{B}$ a *BA of factor relations on A* if B is a set of factor relations,

$$0 = \mathrm{Id}_A, \quad 1 = A^2$$

and $\cdot$ is the intersection of congruence relations. It is not necessary to assume $+$ agrees with the lattice sum in $\Theta(A)$ nor that $\mathscr{B}$ consist of all factor relations on A. We say that $\mathscr{B}$ is a *strong BA of factor relations on A* if, in addition, $+$ is the relative multiplication of congruences.

The Standard Construction. Suppose $\mathscr{B}$ is a BA of factor relations on A. Let X be the Stone space of $\mathscr{B}$, i. e., the set of all maximal ideals of $\mathscr{B}$ with the usual topology. For $x \in X$ let $\lambda(x) = \bigcup x$ (equals the congruence relation generated by x) and let S_x denote $A/\lambda(x)$. Let S be the disjoint union of the sets $S_x (x \in X)$. For notational convenience we identify $(x, a/\lambda(x))$ with $a/\lambda(x)$ in the formation of the disjoint union. Define $\pi : S \to X$ so that $\pi(a/\lambda(x)) = x$. For $a \in A$ define the auxiliary map $r_a : X \to S$ by $r_a(x) = a/\lambda(x)$. We make S a topological space by using $\{r_a(U) : a \in A,\ U$ open subset of $X\}$ as a basis. The topology produced will be the finest one making all the functions r_a continuous.

An isomorphism $h : A \cong \Gamma(X, S)$ where (X, S) is a sheaf is called a *sectional representation of A*.

Theorem. The (X, S, π) constructed above is a sheaf and the map $\xi_{\mathscr{B}} : a \mapsto r_a$ is a sectional representation of A onto $\Gamma(X, S)$. Call $\xi_{\mathscr{B}}$ the *canocical representation associated with the BA $\mathscr{B}$*.

The above theorem will be applied in situations where $\mathscr{B}$ is a complete atomic strong BA of factor relations on A. Throughout the paper, when dealing with such a $\mathscr{B}$, certain letters will have the following fixed meaning. Of course X denotes the Stone space of B and S denotes the sheaf over X constructed above in the standard way. Let J be the set of all isolated points of X; for $x \in J$, x is the principal maximal ideal of $\mathscr{B}$ generated by the dual atom $\lambda(x)$. For $\varphi \in \mathscr{B}$, $N(\varphi) = \{M \in X : \varphi \notin M\}$ and $J(\varphi) = \{M \in J : \varphi \in M\}$. Of course $N(\varphi)$ is a basic open and closed set used to define the topology on X. The J-operation is related to the closure operation on X; the closure $\bar{Y}$ of $Y \subseteq X$ is

$$\bar{Y} = \{M \in X : M \supseteq \bigcap Y\}.$$

In case $\mathscr{B}$ is complete and atomic there is a natural isomorphism between $P(J)$ and $\mathscr{B}$. Explicitly, ν is defined for $Y \subseteq J$ by

$$\nu(Y) = -\bigcap \lambda(Y).$$

Observe that since every element in $\mathscr{B}$ is the meet of all dual atoms containing it, also $\nu(Y) = \bigcap \lambda(J - Y)$. This observation will be used several times.

2. BOOLEAN algebras induced by restricted product decompositions

First we recall the definition of an L-restricted direct product due to HASHIMOTO [3]; cf., also GRÄTZER [2].

Definition 2.1. *Suppose L is an ideal in the BA $P(I)$ and A_i are algebras for $i \in I$. An algebra A' is an L-restricted direct product* of $\langle A_i : i \in I\rangle$ *if*

(i) *A' is a subalgebra of $\prod_{i \in I} A_i$,*

(ii) *$f, g \in A'$ implies $\{i \in I : f(i) \neq g(i)\} = D(f, g) \in L$,*

(iii) *$f \in A'$, $g \in \Pi_{i \in I} A_i$, and $D(f, g) \in L$ implies $g \in A'$.*

Special cases of interest are $L = P(I)$, in which case the L-restricted product A' is $\prod_{i \in I} A_i$, and $L = S_\omega(I)$, i. e., the ideal of all finite subsets of I, in which case the L-restricted product A' is the weak direct product of $\langle A_i : i \in I\rangle$. In the following only product notions intermediate to the weak and strong direct products are considered. That is, we usually assume $L \supseteqq S_\omega(I)$. It is known that this assumption can be made without loss of generality; see GRÄTZER [2].

An *L-restricted product decomposition of A* is an isomorphism $h : A \cong A'$ where A' is an L-restricted direct product of $\langle A_i : i \in I\rangle$.

Our first goal is associate a BA of factor relations with an L-restricted product decomposition. This is done by considering various restricted products formed from a given one.

For $U \subseteqq I$ let $L_U = \{K \cap U : K \in L\}$. Equivalently, $L_U = \{K \subseteqq U : K \in L\}$. Clearly, L_U is an ideal in $P(U)$.

Suppose A' is an L-restricted direct product of $\langle A_i : i \in I\rangle$. For $U \subseteqq I$, define

$$r_U : A' \to \prod_{i \in U} A_i \text{ by}$$

$$r_U(f) = f \upharpoonright U .$$

The image $A'_U = r_U(A')$ of A' is easily seen to be an L_U-restricted direct product of $\langle A_i : i \in U\rangle$. As a convention we let A'_U be *a* one element algebra if $U = 0$. For $U \subseteqq I$ let $r_U \times r_{I-U}$ be the unique homomorphism,

$$r_U \times r_{I-U} : A' \to A'_U \times A'_{I-U},$$

defined for $f \in A'$ by

$$r_U \times r_{I-U}(f) = (r_U(f), r_{I-U}(f)) .$$

Lemma 2.2. *For $U \subseteqq I$, $r_U \times r_{I-U} : A' \cong A'_U \times A'_{I-U}$.*

Proof. It is easily seen that the map is one-one using 2.1 (ii). Now suppose $(f, g) \in A'_U \times A'_{I-U}$. Then $h = f \cup g \in \Pi_{i \in I} A_i$. Choose $e \in A'$. Since

$$D_1 = D(f, r_U(e)) \in L \quad \text{and} \quad D_2 = D(g, r_{I-U}(e)) \in L,$$

we have $D(f \cup g, e) = D_1 \cup D_2 \in L$. By 2.1 $h \in A'$ and $r_U \times r_{I-U}(h) = (f, g)$ as desired.

Remark. If $L = P(I)$, then $L_U = P(U)$; so A'_U is also a full direct product. If $L = S_\omega(I)$, then $L_U = S_\omega(U)$ and A'_U is again the weak direct product in this case.

Suppose $h: A \cong A'$ where A' is an L-restricted direct product of $\langle A_i: i \in I\rangle$. For $U \subseteq I$, define

$$\vartheta_U = \ker(r_{I-U} \circ h)$$

It is useful to observe that $\vartheta_U = \{(a, b) \in A^2: h(a) \restriction I - U = h(b) \restriction I - U\} = \{(a, b) \in A^2: D(h(a), h(b)) \subseteq U\}$.

Theorem 2.3. *The collection $\{\vartheta_U: U \subseteq I\}$ of factor relations of A forms a BA under relative multiplication and intersection. In fact the map $\mu: U \mapsto \vartheta_U$ is an epimorphism of $P(I)$ onto this BA. μ is an isomorphism if each A_i is nontrivial and $L \supseteq S_\omega(I)$.*

Proof. By 2. 2, ϑ_U is a factor relation on A. Clearly $\vartheta_0 = \mathrm{Id}_A$ and $\vartheta_I = A^2$. It is straightforward to verify

(1) for $U, V \subseteq I, \vartheta_U \cap \vartheta_V = \vartheta_{U \cap V}$.

We show

(2) for $U, V \subseteq I, \vartheta_U \mid \vartheta_V = \vartheta_{U \cup V}$.

It is immediate that $\vartheta_U \mid \vartheta_V \subseteq \vartheta_{U \cup V}$. Now suppose $(a, b) \in \vartheta_{U \cup V}$. Let $k = (h(a) \restriction I - U) \cup (h(b) \restriction U) \in \Pi_{i \in I} A_i$ and choose $e \in A$. By 2.1 (ii), $D(h(e), h(a)) \in L$ and $D(h(e), h(b)) \in L$. It easily follows that $D(h(e), k) \in L$ and so, by 2.1 (iii), $k \in A'$. Hence, $k = h(c)$ for some $c \in A$. Using the facts that $h(a)$ and $h(c)$ agree on $I - U$, $h(a)$ and $h(b)$ agree on $I - (U \cup V)$ and that k and $h(b)$ agree on $(I - V) \cap U$, it follows that $a\, \vartheta_U\, c\, \vartheta_V\, b$. Thus $(a, b) \in \vartheta_U \mid \vartheta_V$ as desired.

Each ϑ_U has a complement, namely ϑ_{I-U}. This, together with (1) and (2), implies $\{\vartheta_U: U \subseteq I\}$ is a homomorphic image of the $P(I)$ and a BA under relative multiplication and intersection.

Now suppose $S_\omega(I) \subseteq L$ and each A_i is nontrivial. If $0 \neq U \subseteq I$, choose $i \in U, e \in A$ and define $f, g \in \prod_{i \in I} A_i$ so that $D(h(e), f) = D(h(e), g) = \{i\}$ and $f(i) \neq g(i)$. Since $S_\omega(I) \subseteq L$, $D(f, g) = \{i\} \in L$ and therefore $(f, g) \in \vartheta_U$. Hence $\vartheta_U \neq \mathrm{Id}_A$ and μ is one-one.

Let $\mathscr{B}_h$ denote the BA $\{\vartheta_U: U \subseteq I\}$; we call it the *BA of factor relations of A associated with the L-restricted product decomposition h of A*. From now on we only consider L-restricted product decompositions in which all A_i's are nontrivial. By 2.3, whenever $L \supseteq S_\omega(I)$, $\mu: P(I) \cong \mathscr{B}_h$.

We need some additional properties of the BA $\mathscr{B}_h$.

Lemma 2.4. *Suppose L is an ideal in $P(I)$ with $L \supseteq S_\omega(I)$ and $h: A \cong A'$ where A' is an L-restricted product of nontrivial algebras A_i. Then, for every $x \in X$ (the Stone space of $\mathscr{B} = \mathscr{B}_h$) if $x \supseteq \mu(L)$, then $\lambda(x) = A^2$.*

Proof. The map μ from 2.3 gives an isomorphism between $P(I)$ and $\mathscr{B}_h$. Using μ a maximal ideal M of $\mathscr{B}_h$ yields a maximal ideal $\mu^{-1}(M)$ of $P(I)$ and an ultrafilter $F = \{U \subseteq I : \vartheta_{I-U} \in M\}$ on I. The ultrafilter F induces a congruence relation $\sim_F$ on A', where, by definition,

$$\begin{aligned}\sim_F &= \{(f, g) \in A' \times A' : \{i \in I : f(i) = g(i)\} \in F\} \\ &= \{(f, g) \in A' \times A' : D(f, g) \in \mu^{-1}(M)\}.\end{aligned}$$

Using the definition of $\lambda(M)$, properties of ϑ_U and F, and the definition of $\sim_F$, it follows that $h(\lambda(M)) = \sim_F$ for every maximal ideal M of $\mathscr{B}_h$. Now suppose a maximal ideal M has the property that $\mu^{-1}(M) \supseteq L$. For $f, g \in A'$, 2.1 (ii), implies $D(f, g) \in L \subseteq \mu^{-1}(M)$ so $f \sim_F g$. Hence $\sim_F = A' \times A'$ and since $\sim_F = h(\lambda(M))$, $\lambda(M) = A^2$ as desired.

Lemma 2.5. *Suppose L is an ideal in $P(I)$ with $L \supseteq S_\omega(I)$ and $h: A \cong A'$ where A' is an L-restricted product of nontrivial algebras A_i. Then for every $\psi \in \mu(L)$ and every $f \in {}^{J(-\psi)}A$, $\bigcap_{i \in J(-\psi)} f(i)/\lambda(i) \neq 0$.*

Proof. Assume $\psi \in \mu(L)$, say $\psi = \vartheta_U$ and $f \in {}^{J(-\psi)}A$. The set $\lambda(J(-\psi))$ consists of all dual atoms containing $-\psi = \vartheta_{I-U}$. In view of the isomorphism μ,

(1) $\lambda J(-\psi) = \{\vartheta_{I-\{i\}} : i \in U\}$.

Using (1), f induces a function $\bar{f} \in {}^U A$; namely, for $i \in U$,

$$\bar{f}(i) = f(\lambda^{-1}(\vartheta_{I-\{i\}})).$$

Choose $k \in A'$ and define $\bar{k} \in \prod_{i \in I} A_i$ by

$$\bar{k}(i) = \begin{cases} k(i) & \text{if } i \notin U \\ h(\bar{f}(i))(i) & \text{if } i \in U. \end{cases}$$

Clearly $D(k, \bar{k}) \subseteq U$. Also, $\psi = \vartheta_U \in \mu(L)$ so $U \in L$ and, hence, $D(k, \bar{k}) \in L$. Consequently, 2.1 (iii) implies $\bar{k} \in A'$. Choose $a \in A$ such that $h(a) = \bar{k}$. Now,

(2) for all $x \in J(-\psi)$, $a/\lambda(x) = f(x)/\lambda(x)$.

Let $x \in J(-\psi)$. By (1), $\lambda(x) = \vartheta_{I-\{j\}}$ for some unique $j \in U$. Also note that $\bar{f}(j) = f(x)$ from the definition of $\bar{f}$. Hence, $h(a)(j) = \bar{k}(j) = h(\bar{f}(j))(j) = h(f(x))(j)$; so $h(a) \upharpoonright \{j\} = h f(x) \upharpoonright \{j\}$. This equality is equivalent to $(a, f(x)) \in \vartheta_{I-\{j\}} = \lambda(x)$ which proves (2). The conclusion of 2.5 follows immediately from (2).

The properties of the B A associated with h are summarized below.

Theorem 2.6. For each L-restricted product decomposition h of A with $L \supseteq S_\omega(I)$ (and nontrivial factors) there is associated a B A $\mathscr{B}$ and ideal K of $\mathscr{B}$ with the properties:

(1) $\mathscr{B}$ is a complete, atomic strong B A of factor relations on A,

(2) for $x \in X$, $x \supseteq K$ implies $\lambda(x) = A^2$,

(3) for every $\psi \in K$, every $f \in {}^{J(-\psi)}A$, $\bigcap_{i \in J(-\psi)} f(i)/\lambda(i) \neq 0$

(4) for every $x \in X$, $x \supseteq K$ implies $x \in X - J$.

Proof. Let $\mathscr{B} = \mathscr{B}_h$ and $K = \mu(L)$. Parts (1), (2), (3) follow from 2.3, 2.4, 2.5 respectively. (4) is an immediate consequence of $L \supseteq S_\omega(I)$.

If h is a weak direct product decomposition of A, i. e., $L = S_\omega(I)$, K consists of all finite sums of atoms and thus, $\{x \in X : x \supseteqq K\}$ consist of exactly the nonisolated points of X. If h is a (strong) direct product decomposition of A, i. e., $L = P(I)$, then $K = \mathscr{B}$ so $\{x \in X : x \supseteqq K\}$ is empty.

Theorem 2.6 holds without assuming $L \supseteqq S_\omega(I)$ and all factors are nontrivial. The idea is to let $I' = \{i \in I : A_i$ is nontrivial and $\{i\} \in L\}$ and $h' = r_{I'} \circ h$. Then $h' : A = h'(A)$ where $h'(A)$ is a $L_{I'}$-restricted product of $\langle A_i : i \in I' \rangle$. But $L_{I'} \supseteqq S(I')$ and each A_i is nontrivial for $i \in I'$. If μ' is the isomorphism from 2.3 associated with h' and $U \subseteqq I$, then $\mu(U) = \mu'(U \cap I')$. As a consequence of this, $\mu' : P(I') \cong \mathscr{B}_{h'} = \mathscr{B}_h$ and $\mu(L) = \mu'(L_{I'})$. Thus, 2.6 (1)—(4) still hold.

In 3.6 we will recapture h from $\mathscr{B}_h$. For this we need to know when two decompositions of A are essentially the same.

Suppose $h : A \cong A'$ and $h' : A \cong C'$ where A' is an L-restricted direct product of $\langle A_i : i \in I \rangle$ and C' is an L'-restricted direct product of $\langle C_i : i \in I' \rangle$. We say h *and* h' *are equivalent* if there exists a bijection k between I and I' and isomorphism $m_i : A_i \cong C_{k(i)} (i \in I)$ such that the induced isomorphism $m = \prod_{i \in I} m_i : \prod_{j \in I'} A_i \cong \prod_{j \in I'} C_j$ has the property $m \circ h = h'$. For the record, m is defined for $f \in \prod_{i \in I} A_i$ and $i \in I$ by the condition:

$$m(f)\,(k(i)) = m_i\,(f(i)) .$$

Also observe that the condition $m \circ h = h'$ implies $m : A' \cong C'$ and $k(L) = L'$.

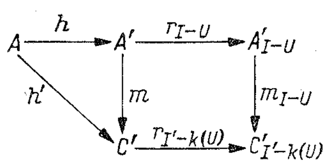
Theorem 2.7. *If* h *and* h' *are equivalent restricted product decompositions, then* $\mathscr{B}_h = \mathscr{B}_{h'}$.

Proof. For $U \subseteqq I$ define $m_U = \prod_{i \in U} m_i : \prod_{i \in U} A_i \to \prod_{j \in k(U)} C_j$ analogous to m.

Because of the coordinatewise definitions of both m and m_U, it is easily seen that

$$(1)\ r_{k(U)}\, m(f) = m_U\,(r_U(f)) \quad \text{for all} \quad f \in \prod_{i \in I} A_i \quad \text{and} \quad U \subseteqq I .$$

From (1) it easily follows that $C'_{k(U)}$ is the image of A'_U under the isomorphism m_U. Also, from (1) and the definition of equivalence the following diagram commutes.



(h, h', m, and m_{I-U} are isomorphisms.)

Hence, $\ker(r_{I-U} \circ h) = \ker(r_{I'-k(U)} \circ h')$ so the B A′ s $\mathscr{B}_h$ and $\mathscr{B}_{h'}$ coincide.

3. Canonical restricted product decompositions

In this section we exploit the sectional representation construction in Section 1 to obtain restricted product decompositions. The idea goes as follows: suppose $\mathcal{B}$ is an atomic strong B A of factor relations on A; now let $h = r_J \circ \xi : A \to \prod_{x \in J} S_x$ where $\xi_{\mathcal{B}} : A \cong \Gamma(X, S)$ is the canonical sectional representation of A induced by $\mathcal{B}$ and r_J is the homomorphism which restricts a continuous section to a function on the set J of isolated points of X. Since $\mathcal{B}$ is atomic, J is dense in X and thus h is one-one. In this case $h : A \cong h(A)$ where $h(A)$ is a subdirect product of the stalks S_x, $x \in J$. In 3.5 we see that for certain BA's $\mathcal{B}$ there is an ideal L in $P(J)$ so that the above $h(A)$ is an L-restricted direct product of $\langle S_x : x \in J \rangle$. The desired BA's are characterized by conditions 2.6 (1)—(4).

Before proceeding we need some auxiliary facts. In the following lemmas we assume that $\mathcal{B}$ is a complete atomic strong BA of factor relations on A. These lemmas establish connections between the topology on X, subsets of J, and elements of $\mathcal{B}$. In addition to the terminology from section 1, we also need the following notation. For $f, g \in \Gamma(X, S)$, let $E(f, g) = \{x \in X : f(x) = g(x)\}$ and $D(f, g) = X - E(f', g')$. We use the same notation for elements of

$$r_J(\Gamma(X, S)); E(f', g') = \{x \in J : f'(x) = g'(x)\}$$

and $D(f', g') = J - E(f', g')$ for $f', g' \in r_J(\Gamma(X, S))$. It will be clear from the context which is meant.

Lemma 3.1. *For a subset U of J, $\bar{U}$ is a clopen subset of X; in fact,*

$$\bar{U} = N(- \cap \lambda\, U).$$

Proof. Since $\lambda(U)$ is a set of dual atoms of $\mathcal{B}$, $\psi = \cap \lambda(U) \in \mathcal{B}$. For $\vartheta \in \mathcal{B}$, $\vartheta \in \cap U$ iff ϑ is contained in every member of $\lambda(U)$ and this is equivalent to $\vartheta \leq \psi = \cap \lambda(U)$. Thus, $\cap U$ is the ideal in $\mathcal{B}$ generated by ψ. Now $x \in \bar{U}$ means, by definition, that $x \supseteq \cap U$ and, by the statement above, this is equivalent to $\psi \in x$. This last condition is equivalent to $x \in N(-\psi)$.

Lemma 3.2. *For a subset U of J, $U = J\,(\cap \lambda(U))$.*

Proof. Clearly $U \subseteq J\,(\cap \lambda(U))$. Suppose $i \in J\,(\cap \lambda(U))$ and $i \notin U$. Then $\cap \lambda(U) \in i$ and $\lambda(i) \notin \lambda(U)$. Thus, $-\lambda(i) \leq \lambda(j)$ for all $j \in U$ and, therefore, $-\lambda(i) \leq \cap \lambda(U) \leq \lambda(i)$ which is a contradiction.

Lemma 3.3. *Suppose $f, g \in \Gamma(X, S)$, $f' = r_J(f)$, $g' = r_J(g)$. Then*

(i) $\overline{E(f', g')} = N(\varphi)$ *where* $\varphi = \cap \lambda\, D(f', g')$.

(ii) $E(f, g) \subseteq \overline{E(f', g')}$.

Proof. (i): by 3.1 and the fact that $-\cap \lambda\, E(f', g') = \varphi$. (ii). $E(f, g)$ is an open subset of X. For $x \in E(f, g)$, there is a clopen neighborhood N of x with $N \subseteq E(f, g)$, say $N = N(\varphi)$. By 3.1 and the fact $\cap \lambda\, J(-\varphi) = -\varphi$, $\overline{N \cap J} = N$. Thus, $N = \overline{N \cap J} \subseteq \overline{E(f, g) \cap J} = \overline{E(f', g')}$ so $x \in \overline{E(f', g')}$ as desired.

Lemma 3.4. *Same hypothesis as 3.3 and, in addition, suppose L' is an ideal in $P(J)$ and $\varphi = \bigcap \lambda D(f', g')$. Then the following are equivalent:*

(i) $D(f', g') \in L'$

(ii) $-\varphi \in \nu(L')$

(iii) *every maximal ideal containing $\nu(L')$ belongs to $N(\varphi)$.*

Proof. (i) $\Leftrightarrow$ (ii). By the definition of φ and ν, $\nu(D(f', g') = -\varphi$. Thus, $D(f', g') \in L'$ iff $-\varphi \in \nu(L')$.

(ii) $\Rightarrow$ (iii) is obvious. (iii) $\Rightarrow$ (ii) since $\nu(L')$ is the intersection of all maximal ideals containing $\nu(L')$.

Theorem 3.5. *If $\mathscr{B}$ is a BA and K is an ideal in $\mathscr{B}$ with the properties 2.6 (1)—(4), then $h = r_J \circ \xi\colon A \cong h(A)$ where $h(A)$ is an L-restricted direct product of $\langle S_x : x \in J\rangle$ for some L in $P(J)$ with $L \supseteq S_\omega(J)$. Actually, $\nu(L) = K$ where ν is the natural isomorphism $\nu\colon P(J) \cong \mathscr{B}$ from section 1. For $\mathscr{B}$ and K with 2.6 (1)—(4), we refer to the above construction as the canonical restricted product decomposition of A associated with $(\mathscr{B}, K)$.*

Proof. Suppose $\mathscr{B}$ and K satisfy the hypothesis. From 2.6 (1) and section 1, $\nu\colon P(J) \cong \mathscr{B}$. Let L be the ideal in $P(J)$ such that $\nu(L) = K$. We are to verify that $A' = h(A)$ satisfies the conditions in 2.1 for $\langle S_x : x \in J\rangle$. Clearly 2.1 (i) holds.

Next we check 2.1 (ii). Suppose $f', g' \in A'$. Choose $f, g \in \Gamma(X, S)$ such that $r_J(f) = f'$ and $r_J(g) = g'$. Condition 2.6 (2), 3.3 (ii) and 3.3 (i) give $\overline{\nu(L)} \subseteq E(f, g) \subseteq \overline{E(f', g')} = N(\varphi)$ where $\varphi = \bigcap \lambda D(f', g')$. From 3.4 we conclude $D(f', g') \in L$ and, thus, 2.1 (ii) holds.

For 2.1 (iii), assume $f' \in A'$, $g' \in \prod_{x \in J} S_x$ and $D(f', g') \in L$. Choose $f \in \Gamma(X, S)$ such that $r_J(f) = f'$. Let $\varphi = \bigcap \lambda D(f', g')$. By 3.4, $-\varphi \in K$ since $D(f', g') \in L$. We are going to apply 2.6 (3) with $\psi = -\varphi$.

For each $j \in J(-\psi) = J(\varphi)$, $g'(j) \in S_j = A/\lambda(j)$. By the axiom of choice choose $g'' \in {}^{J(\varphi)}A$ such that $g'(j) = g''(j)/\lambda(j)$ for every $j \in J(\varphi)$. Applying 2.6 (3) with $-\varphi$ and g'', there is an $a \in A$ such that $a/\lambda(j) = g''(j)/\lambda(j) = g'(j)$ for every $j \in J(\varphi)$. Thus, $\xi(a) \restriction J(\varphi) = g' \restriction J(\varphi)$.

Define g as $g = (f \restriction N(\varphi)) \cup (\xi(a) \restriction N(-\varphi))$. $g \in \Gamma(X, S)$ since both f and $\xi(a)$ belong to $\Gamma(X, S)$.

By 3.3 (i), $N(\varphi) = \overline{E(f', g')}$ and by 3.1, with $U = D(f', g')$, $N(-\varphi) = \overline{D(f', g')}$. Thus, $N(\varphi) \cap J = E(f', g')$ and $N(-\varphi) \cap J = D(f', g')$. Also, by 3.2, $J(\varphi) = D(f', g')$. Now,

$$\begin{aligned} r_J(g) = g \restriction J &= (f \restriction E(f', g')) \cup (\xi(a) \restriction D(f', g')) \\ &= (f' \restriction E(f', g')) \cup (g' \restriction D(f', g')) \\ &= g'. \end{aligned}$$

Since $g \in \Gamma(X, S)$, $g' \in A$ and so 2.1 (iii) holds.

Hence, $h(A) = A'$ is an L-restricted direct product of $\langle S_x : x \in J\rangle$. The proof of 3.5 is completed with the observation that 2.6 (4) implies $L \supseteqq S_\omega(J)$.

We conclude this section by showing that every restricted product decomposition can be produced by the method of 3.5.

Theorem 3.6. *Suppose $L \supseteqq S_\omega(I)$. An L-restricted product decomposition h of A is equivalent to the canonical restricted product decomposition of A associated with $(\mathscr{B}_h, \mu(L))$.*

Proof. Suppose $h: A \cong A'$ where A' is an L-restricted direct product of nontrivial algebras $\langle A_i : i \in I\rangle$. Since $S_\omega(I) \subseteq L$ and A_i is nontrivial, $A'_{\{i\}} = A_i$. Since $\mu: P(I) \cong \mathscr{B}_h$ and $\nu: P(J) \cong \mathscr{B}_h$ there is a natural one-one map k of I onto J. This is defined as follows. For $i \in I$, let $k(i) = j$ where j is the principal maximal ideal generated by the dual atom $\mu(I - \{i\})$. Also, recall, $\mu(I - \{i\}) = \vartheta_{I-\{i\}} = \ker(r_{\{i\}} \circ h)$. From these definitions and the fact that $A_{\{i\}} = A_i$ we see there are natural isomorphisms m_i for $i \in I$:

$$m_i : A_i = A'_{\{i\}} \cong A/\mu(I - \{i\}) = A/\lambda(k(i)) = S_{k(i)}.$$

A direct definition of $m_i: A_i \cong S_{k(i)}$ is given by $m_i(h(a)(i)) = a/\lambda(k(i))$ for all $a \in A$.

Hence, $m(h(a))(k(i)) = m_i(h(a)(i)) = a/\lambda(k(i))$ where $m = \prod_{i \in I} m_i$ (see definition of equivalent), $a \in A$ and $i \in I$. Thus, $m \circ h = h'$ where $h' = r_J \circ \xi$ is the canonical restricted product decomposition of A associated with $(\mathscr{B}_h, \mu(L))$. Thus, h and h' are equivalent.

4. Characterization results

The previous sections dealt with connections between restricted product decompositions of an algebra and BA's with distinguished ideals. Now we put them together.

Prior to 2.7 we defined the equivalence of two restricted product decompositions of A; let $[h]$ denote the class of all decompositions equivalent to h. Let $h_{(\mathscr{B}, K)}$ denote the canonical restricted product decomposition of A associated with $(\mathscr{B}, K)$. We say that an L-restricted product decomposition h of A is *nontrivial* if $L \supseteqq S_\omega(I)$ and each A_i is a nontrivial algebra.

Theorem 4.1. *There is a one-one correspondence between the equivalence classes of nontrivial restricted product decompositions of A and pairs $(\mathscr{B}, K)$ of BA's with distinguished ideals satisfying 2.6 (1)—(4). The correspondence $[h] \mapsto (\mathscr{B}_h, \mu(L))$ is the map one way and its inverse is $(\mathscr{B}, K) \mapsto [h_{(\mathscr{B}, K)}]$.*

Proof. By 2.3, 2.6, and 2.7 $[h] \mapsto (\mathscr{B}_h, \mu(L))$ is a well defined function between the sets involved. By 3.5 $(\mathscr{B}, K) \mapsto [h_{(\mathscr{B}, K)}]$ is defined and 3.6 says it is onto (or equivalently, $[h] \mapsto (\mathscr{B}_h, \mu(L))$ is one-one). It remains to show $[h] \mapsto (\mathscr{B}_h, \mu(L))$

is onto. It is enough to assume $(\mathscr{B}, K)$ satisfies 2.6 (1)—(4) and show the pair associated with $h_{(\mathscr{B}, K)}$ is $(\mathscr{B}, K)$. Let $\mathscr{B}_h$ denote the BA of factor relations associated with $h = h_{(\mathscr{B}, K)}$. Since both $\mathscr{B}$ and $\mathscr{B}_h$ are complete atomic strong BA's of factor relations to show $\mathscr{B} = \mathscr{B}_h$ it is enough to show they have the same dual atoms. The set of dual atoms of $\mathscr{B}$ is $\{\lambda(i) : i \in J\}$ while the set of dual atoms of $\mathscr{B}_h$ is $\{\vartheta_{J-\{i\}} : i \in J\}$.

Now, for $a, b \in A$ and $i \in J$,

$$\begin{aligned} (a, b) \in \vartheta_{J-\{i\}} \text{ iff } & D\,(h(a), h(b)) \subseteq J - \{i\} \\ \text{iff } & h(a)\,(i) = h(b)\,(i) \\ \text{iff } & a/\lambda(i) = b/\lambda(i) \\ \text{iff } & (a, b) \in \lambda(i) \end{aligned}$$

Thus, $\vartheta_{J-\{i\}} = \lambda(i)$ and, therefore, $\mathscr{B} = \mathscr{B}_h$.

Now, μ (from 2.3) and ν are both isomorphisms from $P(J)$ onto $\mathscr{B}_h = \mathscr{B}$ which agree on the dual atoms since $\vartheta_{J-\{i\}} = \lambda(i)$. Thus, $\mu = \nu$ and therefore $K = \mu(L)$ as desired.

If we fix the product notion under consideration, i. e., consider only a "fixed" L, then reference to the ideal K disappears from the characterization and we are left with correspondences between decompositions and certain BA's. We give such specializations for weak and strong direct products.

Theorem 4.2. *There is a one-one correspondence between equivalence classes of nontrivial weak direct product decompositions of A and complete, atomic strong BA's $\mathscr{B}$ of factor relations on A such that $\lambda(x) = A^2$ for every nonisolated $x \in X$. The correspondence takes $[h] \mapsto \mathscr{B}_h$ and its inverse $\mathscr{B} \mapsto [h_{(\mathscr{B}, K)}]$ where K is the ideal of all finite sums of atoms in $\mathscr{B}$.*

Proof. It is enough to verify that the correspondences induced by 4.1 behave properly. For $[h] \mapsto \mathscr{B}_h$ this follows from the remark following 2.6. Now, suppose $\mathscr{B}$ is a complete atomic BA of factor relations on A such that $\lambda(x) = A^2$ for every nonisolated $x \in X$. Let K denote the ideal of all finite sums of atoms in $\mathscr{B}$. It is enough to show that $(\mathscr{B}, K)$ satisfies 2.6 (1)—(4). Note that $\{x \in X : x \supseteq K\} = X - J$ so 2.6 (4) holds; also, 2.6 (1), (2) clearly hold by hypothesis. For $\psi \in K$, $-\psi$ is a sum of a confinite set of atoms. Thus, $J(-\psi)$ consist of a finite number of isolated points and hence 2.6 (3) holds. We are finished by 3.5.

Theorem 4.3. *There is a one-one correspondence between equivalence classes of nontrivial (strong) direct product decompositions of A and complete atomic strong BA's $\mathscr{B}$ of factor relations on A with the property: for every $f \in {}^J A$, $\bigcap_{i \in J} f(i)/\lambda(i) \neq 0$. The correspondence is given by $[h] \mapsto \mathscr{B}_h$ and its inverse is given by $\mathscr{B} \mapsto [h_{(\mathscr{B}, \mathscr{B})}]$.*

Proof. It is enough to see that the correspondences in 4.1, when restricted map objects where they belong. For a nontrivial direct product decomposition h, $\mu(L) = \mathscr{B}$ so 2.6 (3) holds with $\psi = 1$; thus, $\mathscr{B}_h$ has the desired properties. It remains to show that $(\mathscr{B}, K)$ satisfies 2.6 (1)—(4) where $\mathscr{B}$ has the properties above and $K = \mathscr{B}$. By the choice of K, 2.6 (2), (4) hold vacously; 2.6 (1) is clear and 2.6 (3) follows easily from the assumptions about $\mathscr{B}$. By 3.5 the proof is finished.

5. Direct products

In the preceding sections an elaborate framework was developed for discussing and comparing various types of decompositions of an algebra. In this section notions are specialized to direct product decompositions in order to use the correspondence in 4.3 to gain insight into properties of direct products. Proofs will be omitted since they are easy and in many cases just a translation of facts between two isomorphic posets.

A result of BIRKHOFF-HASHIMOTO see [3]) characterizes direct product decompositions of an algebra A in terms of special sequences of congruence relations on A. In terms of this characterization two direct product decompositions are equivalent (in the sense defined prior to 2.7) if and only if the two corresponding sequences of congruence relations are the same. Likewise, the other properties considered below such as the refinement property and the unique total decomposition property also correspond to the strict formulation in terms of congruence relations.

Let $F(A) = \{\mathscr{B}: \mathscr{B}$ is a strong BA of factor relations on $A\}$ and let $F_s(A) = \{\mathscr{B} \in F(A): \mathscr{B}$ is complete, atomic and $0 \neq \bigcap_{i \in J} f(i)/\lambda(i)$ for all $f \in^J A\}$. Both $F(A)$ and $F_s(A)$ are posets under the subalgebra relation and $F_s(A) \subseteq F(A)$ as posets. Actually, $F(A)$ is somewhat nicer than $F_s(A)$ in general. $F(A)$ is always an $\bigcap$-semilattice with 0. Of course $F(A) = F_s(A)$ if A is finite.

Let $D_s(A) = \{[h]$: h is a nontrivial direct product decomposition of $A\}$. Suppose $h: A \cong \prod_I A_i$ and $h': A \cong \prod_{I'} C_j$ are direct decompositions of A. We say that h' is a *refinement* of h if there is a k from I' onto I and for each $i \in I$ there are isomorphisms $m_i: A_i \cong \prod_{j \in k^{-1}(i)} C_j$ such that $m \circ h = h'$ where $m = \prod_I m_i: \prod_I A_i \cong \prod_{I'} C_j$. Refinement is a quasi-ordering of decompositions and induces a partial ordering on $D_s(A)$: $[h] \leqq [h']$ iff h' is a refinement of h. When $D_s(A)$ is considered as a poset it is with this partial ordering.

Theorem 5.1. The bijections between $F_s(A)$ and $D_s(A)$ from 4.3 are isomorphisms of posets.

Corollary 5.2. *A has the strict refinement property if and only if $F_s(A)$ is directed.*

Recall that a direct product decomposition $h: A \cong \prod_I A_i$ is *total* if each A_i is directly indecomposable.

Theorem 5.3. *If $h: A \cong \prod_I A_i$ is total, then $\mathscr{B}_h$ is maximal in $F(A)$.*

The following is a consequence of 5.3 and 5.1.

Corollary 5.4. (1) *The isomorphism $D_s(A) \cong F_s(A)$ gives a one-one correspondence between total decompositions of A and maximal elements of $F_s(A)$.* (2) *A maximal element in $F_s(A)$ is also maximal in $F(A)$.*

Let us reexamine the isomorphism $F_s(A) \cong D_s(A)$. It involved two steps. First, $\mathscr{B} \in F_s(A)$ was associated with a sectional representation $\xi_{\mathscr{B}}: A \cong \Gamma(X, S)$. Secondly, $\xi_{\mathscr{B}}$ was associated with a direct decomposition by composing it with a restriction map. Now the first construction can always be made for any $\mathscr{B} \in F(A)$ while the second cannot. Actually the bijections between $D_s(A)$ and $F_s(A)$ in 4.3 can be reformulated to give a correspondence between direct product decompositions and certain sectional representations; namely, $h \mapsto \xi_{\mathscr{B}_h}$. (A similar reformulation can be made for 4.1.)

The above remarks and 5.4 suggest the following definition. A sectional representation $\xi_{\mathscr{B}}: A \cong \Gamma(X, S)$ is *total* if $\mathscr{B}$ is a maximal element of $F(A)$. Corollary 5.4 shows that a direct product decomposition of A into indecomposable factors gives rise to a total representation. However, for infinite algebras, total direct product decompositions may not exist. In contrast to this a ZORN's lemma argument shows that total sectional representations always exist.

Knowledge of the relationship between maximal elements of $F(S)$ and $F_s(A)$ should help explain the pathologies concerning the existance of total direct product decompositions.

An algebra A is said to have a *unique total decomposition* (in the strict sense) if A has a total direct product decomposition and any two total direct decompositions are equivalent (i. e., give the same element of $D_s(A)$).

Let $F\,R(A)$ denote the set of all factor relations on A.

Theorem 5.5. *The following are equivalent.*

(i) *A has a unique total decomposition,*

(ii) *$F_s(A)$ has a top element,*

(iii) *$F\,R(A) \in F_s(A)$,* i. e., *$FR(A)$ is a complete, atomic strong BA and $0 \neq \bigcap_{i \in J} f(i)/\lambda(i)$ for all $f \in {}^J A$.*

Corollary 5.6. *For A finite, A has a unique total decomposition if and only if $FR\,(A)$ is a strong BA.*

The two results above are related to results of TARSKI [5] (see Appendix B).

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