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ELEMENTARY PROPERTIES OF STUCTURES OF SECTION

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Topological algebraic representation theorems of the following form are known for various classes $\mathcal{K}$ of algebraic structures (cf., [1, 2, 4, 6, 9, 11, 13]). A subclass $\mathfrak{M}$ of $\mathcal{K}$ is chosen and its members regarded as known; then $\mathcal{K}$ is shown to consist of all structures isomorphic to structures $\Gamma(X, \mathfrak{S})$ of all continuous sections of sheaves where the stalks belong to $\mathfrak{M}$ and the base space X to a suitable class of topological spaces. $\Gamma(X, \mathfrak{S})$ is a subset of the product $\prod_{x \in X} \mathfrak{S}_x$ where $\mathfrak{S}_x$ is the stalk over $x \in X$; if X has the discrete topology, $\Gamma(X, \mathfrak{S})$ is the full product. The object of this note is to point out that there are situations where the topology on X is not discrete but where an analogue of the Feferman-Vaught results [7] hold, i.e., an elementary property of $\Gamma(X, \mathfrak{S})$ can be reduced to properties of the stalks of $\mathfrak{S}$ and a property of a suitable structure of subsets of X.

Preliminary definitions are given in section 1. The main result (Theorem 1.1) is a generalization of the basic result 3.1 in [7] on generalized products. Some consequences of 1.1 are also given. In section 2 Theorem 1.5 is combined with known representation results of the form indicated above to establish the decidability of the theory of m-rings and the theory of Post algebras of order m.

The results sketched in this paper were developed during 1970 and finally announced in [5]. Since that time the author has learned that C. J. Ash has also obtained a decidability proof for Post algebras in addition to other interesting results (cf. [3]). The author is grateful to Angus Macintyre for pointing out an error in the original manuscript. The author had originally asserted that condition (C) held for any sheaf of models of a model-complete theory. As Macintyre observes in [12] the result should be for positively model-complete theories. Macintyre also observed an overlap between the results contained here for constant sheaves and the results [15, 16, 17, 18] on generalized limit powers and reduced powers.

1. Let L be a first order language. A *sheaf of L-structures* is a triple $(X, \mathfrak{S}, \pi)$ where (i) X and $\mathfrak{S}$ are topological spaces, (ii) π is a local homeomorphism from $\mathfrak{S}$ onto X, (iii) $\pi^{-1}x = \mathfrak{S}_x$ is the universe of an L-structure $\mathfrak{S}_x$ for each $x \in X$, (iv) for each individual constant a of L the map $X \to \mathfrak{S}$ that sends $x \in X$ to the value of a in $\mathfrak{S}_x$ is continuous, (v) for each n-ary operation symbol f of L, $n > 0$, the map sending $(s_0, \cdots, s_{n-1}) \to f(s_0, \cdots, s_{n-1})$ is a continuous function from $\bigcup_{x \in X} \mathfrak{S}_x^n$ (with the relative product topology inherited from $\mathfrak{S}^n$) into $\mathfrak{S}$, and (vi) for each n-ary relation symbol R of L, $\{(s_0, \cdots, s_{n-1}) \in \bigcup_{x \in X} \mathfrak{S}_x^n : Rs_0, \cdots, s_{n-1}\}$ is a closed-open subset of $\bigcup_{x \in X} \mathfrak{S}_x^n$. The structures $\mathfrak{S}_x$ are called the *stalks* of the sheaf. We assume throughout the paper that X is a Boolean space.

A *section* of a sheaf $(X, \mathfrak{S}, \pi)$ of L-structures is a continuous map $\sigma: X \to \mathfrak{S}$ such that $\pi\sigma$ is the identity on X. The set of all sections is denoted by $\Gamma(X, \mathfrak{S})$.

[1] The author is grateful to the referee for his many valuable suggestions.

$\Gamma(X, \mathfrak{S})$ can be made into an L-structure by using the operations and relations induced from the product $\Pi_{x \in X} \mathfrak{S}_x$.

Let L_B be any first order language that includes the language for BA's with symbols $+, \cdot, -, 0, 1$. For any Boolean space X, $\mathfrak{C}(X)$ denotes an L_B-structure whose underlying Boolean part is the BA $\mathfrak{C}(X)$ of all closed-open subsets of X. In section 2 we will be particularly interested in the situation where X is equipped with a sequence $U_0, \cdots, U_{k-1}$ of distinguished open subsets. In this case $\mathfrak{C}(X)$ denotes the structure $\langle \mathfrak{C}(X), I_j \rangle_{j<k}$ where I_j is the ideal corresponding to U_j.

The following concepts are similar to those introduced in [7] for products.

If θ is an L-formula with free variables among the first n variable, $(X, \mathfrak{S}, \pi)$ a sheaf of L-structures, and $\sigma \in \Gamma(X, \mathfrak{S})^m$ where $n \leq m \leq \omega$, then set

$$K_\theta^{X,\mathfrak{S}}(\sigma) = \{x \in X : \mathfrak{S}_x \models \theta[\sigma(x)]\}.$$

Notice that if θ is a sentence, $K_\theta^{X,\mathfrak{S}}(\sigma)$ does not depend on σ; we denote the set by $K_\theta^{X,\mathfrak{S}}$ in this case.

A sequence $\zeta = \langle \Phi, \theta_0, \cdots, \theta_{m-1} \rangle$ is called an *acceptable sequence* if each θ_i is an L-formula and Φ is an L_B-formula with free variables among the first m variables. A variable of L is called a *free variable of* ζ if it is free in at least one of the θ_i's. An acceptable sequence is called *standard* if its free variables form an initial sequence of variables. An acceptable sequence ζ is called a *partitioning* sequence if the formula $\theta_0 \vee \cdots \vee \theta_{m-1}$ and the formulas $\neg(\theta_i \wedge \theta_j)$ for $i < j < m$ are logically valid.

The elementary facts about these concepts can be derived in the same way they are established in [7] for products. In particular, a relation can be induced on $\Gamma(X, \mathfrak{S})$ in the following way. If $(X, \mathfrak{S}, \pi)$ is a sheaf of L-structures, $\mathfrak{C}(X)$ an L_B-structure, and $\zeta = \langle \Phi, \theta_0, \cdots \theta_{m-1} \rangle$ a standard acceptable sequence with n free variables, then an n-ary relation is defined on $\Gamma(X, \mathfrak{S})$ by

$$Q_\zeta^{\mathfrak{C}(X),\mathfrak{S}} = \{\sigma \in \Gamma(X, \mathfrak{S})^n : \mathfrak{C}(X) \models \Phi[K_{\theta_0}{}^{X,\mathfrak{S}}(\sigma), \cdots, K_{\theta_{m-1}}{}^{X,\mathfrak{S}}(\sigma)]\}.$$

In the above definition, for $\sigma \in Q_\zeta^{\mathfrak{C}(X),\mathfrak{S}}$ it is understood that each $K_{\theta_i}{}^{X,\mathfrak{S}}(\sigma)$ must be closed-open. To guarantee this always happens we restrict ourselves to sheaves $(X, \mathfrak{S}, \bar{\mu})$ with the property

(C) for every L-formula θ and every $\sigma \in \Gamma(X, \mathfrak{S})^\omega$, $K_\theta^{X,\mathfrak{S}}(\sigma)$ is a closed-open subset of X.

A theory T is *positively model-complete* if, relative to T, every formula is equivalent to a positive existential formula. For the applications in section 2, as well as the realization that the results below have some content, it is important to observe that condition (C) holds in any sheaf of models of a positively model-complete theory (cf., Macintyre [12]). To see this, first notice that the continuity properties for sheaves imply $K_\theta^{X,\mathfrak{S}}(\sigma)$ is open for any atomic formula θ; hence it is also open for any positive existential formula θ. Relative to a positive model-complete theory both a formula and its negative are equivalent to positive existential formulas. Thus, $K_\theta^{X,\mathfrak{S}}(\sigma)$ and its complement are both open as desired.

For a sheaf $(X, \mathfrak{S}, \pi)$ satisfying condition (C), an L_B-structure $\mathfrak{C}(X)$, and a set Ω of standard acceptable sequences, let $P_\Omega(\mathfrak{C}(X), \mathfrak{S})$ denote the structure $\langle \Gamma(X, \mathfrak{S}), Q_\zeta^{\mathfrak{C}(X), \mathfrak{S}} \rangle_{\zeta \in \Omega}$. We call $P_\Omega(\mathfrak{C}(X), \mathfrak{S})$ a *generalized structure of sections* (*sectional structure* for short) and denote its language by L_Ω. Notice that whenever $P_\Omega(\mathfrak{C}(X), \mathfrak{S})$ exist it is taken for granted that $(X, \mathfrak{S}, \pi)$ satisfies (C).

If f is an n-ary operation symbol of L, $\zeta = \langle X_0 = 1, fv_0 \cdots v_{n-1} = v_n \rangle$ is a standard acceptable sequence. For any sheaf $(X, \mathfrak{S}, \pi)$ of L-structures, $Q_\zeta^{\mathfrak{C}(X), \mathfrak{S}}$ is the n-ary operation (defined as an $(n + 1)$-ary relation) on $\Gamma(X, \mathfrak{S})$ that is defined pointwise from the operation f on each stalk. In a similar way a relation symbol of L can be assigned to a standard acceptable sequence ζ so that Q_ζ is just the relation on $\Gamma(X, \mathfrak{S})$ inherited from the product of the stalks. Let Ω_0 be the set of all standard acceptable sequences that correspond to symbols in L. $P_{\Omega_0}(\mathfrak{C}(X), \mathfrak{S})$ is just the familiar L-structure of all sections of the sheaf $(X, \mathfrak{S}, \pi)$ mentioned earlier. Whenever Ω includes Ω_0, it is convenient to regard L_Ω as an extension of L.

Whenever all the stalks of a sheaf are the same we have an analogue of "generalized power". For an L-structure $\mathfrak{A}$, considered with the discrete topology, and a Boolean space X, let $\mathfrak{S} = X \times \mathfrak{A}$ with the product topology and $\pi: \mathfrak{S} \to X$ the projection. The sheaf $(X, \mathfrak{S}, \pi)$ is called the *constant* $\mathfrak{A}$-*sheaf* over X. $\Gamma(X, \mathfrak{S})$, denoted by $\Gamma(X, \mathfrak{A})$ in this case, with the pointwise definition of operations and relations is just isomorphic to the structure of all continuous functions from X into $\mathfrak{A}$. For $\mathfrak{A}$ finite, the structures $\Gamma(X, \mathfrak{A})$ are known as Boolean extensions of $\mathfrak{A}$ (cf., [8], [10]). Sectional structures associated with constant $\mathfrak{A}$-sheafs are denoted as $P_\Omega(\mathfrak{C}(X), \mathfrak{A})$.

Suppose $\mathcal{K}$ is an isomorphism closed class of $\mathfrak{L}_B$-structures and $\mathfrak{M}$ is a class of $\mathfrak{L}$-structures with the property that every sheaf $(X, \mathfrak{S}, \pi)$ of structures in $\mathfrak{M}$ where $\mathfrak{C}(X) \in \mathcal{K}$ satisfies (C). Let $P_\Omega(\mathcal{K}, \mathfrak{M})$ denote the class of all $P_\Omega(\mathfrak{C}(X), \mathfrak{S})$ where $\mathfrak{C}(X) \in \mathcal{K}$ and each stalk of $\mathfrak{S}$ belongs to $\mathfrak{M}$. If $\mathfrak{M} = \{\mathfrak{A}\}$, we shorten the notation to $P_\Omega(\mathcal{K}, \mathfrak{A})$.

The main result dealing with generalized structures of sections is the following.

THEOREM 1.1 *There is an effective procedure that assigns to each L_Ω-formula ϕ an acceptable (partitioning) sequence $\zeta = \langle \Phi, \theta_0, \cdots, \theta_{m-1} \rangle$ with the "same" free variables as ϕ such that for any sheaf $(X, \mathfrak{S}, \pi)$ satisfying (C), any L_B-structure $\mathfrak{C}(X)$, and $\sigma \in \Gamma(X, \mathfrak{S})^\omega$*

$$P_\Omega(\mathfrak{C}(X), \mathfrak{S}) \models \phi[\sigma] \quad \text{iff} \quad \mathfrak{C}(X) \models \Phi[K_{\theta_0}(\sigma), \cdots, K_{\theta_{m-1}}(\sigma)].$$

Consequently, if ϕ is a sentence, so are $\theta_0, \cdots, \theta_{m-1}$ and thus ϕ holds in $P_\Omega(\mathfrak{C}(X), \mathfrak{S})$ iff $\mathfrak{C}(X) \models \Phi[K_{\theta_0}, \cdots, K_{\theta_{m-1}}]$.

Proof. We proceed by induction on ϕ. An atomic formula $Q_\zeta v_{i_1} \cdots v_{i_n}$ is, by definition, related to an acceptable sequence ζ in such a way that the theorem holds. If ϕ is assigned to $\langle \Phi, \theta_0, \cdots, \theta_{m-1} \rangle$, then $\neg\phi$ is assigned to $\langle \neg\Phi, \theta_0, \cdots, \theta_{m-1} \rangle$ and if, in addition, ϕ' is assigned to $\langle \Phi', \theta_0', \cdots, \theta_{K-1}' \rangle$, then $\phi \wedge \phi'$ is assigned to $\langle \Phi \wedge \Phi'(X_m, \cdots, X_{m+K-1}), \phi_0, \cdots, \theta_{m-1}, \theta_0', \cdots, \theta_{K-1}' \rangle$. (cf., proof of 3.1 in [7]).

Before treating the existential quantifier, observe that a partitioning sequence $\langle \Phi', \theta_0', \cdots, \theta_{m'-1}' \rangle$ can be associated (effectively) with any acceptable sequence $\langle \Phi, \theta_0, \cdots, \theta_{m-1} \rangle$ so that they have the same free variables and that

$$\mathfrak{C}(X) \models \Phi[K_{\theta_0}(\sigma), \cdots] \quad \text{iff} \quad \mathfrak{C}(X) \models \Phi'[K_{\theta'_0}(\sigma), \cdots]$$

for all $(X, \mathfrak{S}, \pi)$, $\mathfrak{C}(X)$, and σ. To do this, choose the θ_i''s so that the $K_{\theta'_i}(\sigma)$'s will be the atoms of the BA generated by the $K_{\theta_j}(\sigma)$'s and choose Φ' as the restatement of Φ in terms of these atoms.

Now suppose $\phi = \exists v_k \phi'$ where ϕ' corresponds to a partitioning sequence $\zeta' = \langle \Phi', \theta_0', \cdots, \theta_{m-1}' \rangle$ (use the above remark) with the property of the theorem. Let $\zeta = \langle \Phi, \theta_0, \cdots, \theta_{m-1} \rangle$ where $\theta_j = \exists v_k \theta_j'$ for $j < m$ and $\Phi(X_0, \cdots, X_{m-1})$ is an L_B-formula that says there exist elements $Y_0, \cdots, Y_{m-1}$ such that (a) $Y_0 + \cdots + Y_{m-1} = 1$ and $Y_i \cdot Y_j = 0$ for all $i < j < m$, (b) $Y_i \leq X_i$ for all $i < m$, and (c) $\Phi'[Y_0, \cdots, Y_{m-1}]$ holds. We claim ζ satisfies the theorem for ϕ. For $\sigma \in \Gamma(X, \mathfrak{S})^\omega$ and $\tau \in \Gamma(X, \mathfrak{S})$ let σ_τ^k be the sequence obtained by replacing σ_k by τ in σ. Now, if $P_\Omega(\mathfrak{C}(X), \mathfrak{S}) \models (\exists v_k \phi')[\sigma]$, there exist $\tau \in \Gamma(X, \mathfrak{S})$ such that $\mathfrak{C}(X) \models \Phi'[K_{\theta'_0}(\sigma_\tau^k), \cdots]$. By (C) and the choice of ζ' it follows that $\mathfrak{C}(X) \models \Phi[K_{\theta_0}(\sigma), \cdots]$ when we choose $Y_i = K_{\theta'_i}(\sigma_\tau^k)$ for $i < m$. For the converse, assume $\mathfrak{C}(X) \models \Phi[K_{\theta_0}(\sigma), \cdots]$. Then there are closed-open subsets $Y_0, \cdots, Y_{m-1}$ of X satisfying (a)–(c) with $X_i = K_{\theta_i}(\sigma)$, $i < m$. For each $j < m$, by (b) and the definition of $K_{\exists v_k \theta_j}(\sigma)$, Y_j is covered by a family of closed-open sets of the form $K_{\theta'_j}(\sigma_\tau^k)$. Since Y_j is zero dimensional and compact, there is a partition $Z_{j1}, \cdots Z_{jP_j}$ of Y_j into closed-open subsets and sections $\tau_{j1}, \cdots, \tau_{jP_j}$ such that

(1) $Z_{ji} \subseteq K_{\theta'_j}(\sigma_{\tau_{ji}}^k)$ for all i, j.

Define μ on X such that $\mu \restriction Z_{ji} = \tau_{ji} \restriction Z_{ji}$ for all i, j. The set of Z_{ji}'s partition X so $\mu \in \Gamma(X, \mathfrak{S})$. By the definition of μ and (1),

(2) $Y_j \subseteq K_{\theta'_j}(\sigma_\mu^k)$ for each $j < m$.

Since the sets $Y_0, \cdots, Y_{m-1}$ partition X (by (a)) and the sets $K_{\theta'_0}(\sigma_\mu^k), \cdots, K_{\theta'_{m-1}}(\sigma_\mu^k)$ also partition X (since ζ' is a partitioning sequence), (2) implies $Y_j = K_{\theta'_j}(\sigma_\mu^k)$ for all $j < m$. Hence, (c) and the induction assumption imply $P(\mathfrak{C}(X), \mathfrak{S}) \models \phi'[\sigma_\mu^k]$, i.e., σ satisfies $\exists v_k \phi'$ in $P(\mathfrak{C}(X), \mathfrak{S})$ as desired.

The last part of the above proof illustrates the technique of blowing up local properties of sections to global ones. (cf., [13].)

The fact that 1.1 includes (and extends) the main Theorem 3.1 of [7] can be seen as follows. Suppose $\langle \mathfrak{A}_j : j \in I \rangle$ is a sequence of L-structures. Let X be the space of all ultrafilters on I and $(X, \mathfrak{S}, \pi)$ the sheaf over X with stalks S_F, for $F \in X$, equal the ultraproduct $\Pi \mathfrak{A}_i / F$. For each $a \in \Pi_{i \in I} \mathfrak{A}_i$ define $\sigma_a \in \Gamma(X, \mathfrak{S})$ by $\sigma_a(F) = a/F$ for each $F \in X$. The function that assigns a to σ_a gives an isomorphism of a generalized product of $\langle \mathfrak{A}_i : j \in I \rangle$ in the sense of [7] with a sectional structure $P_\Omega(\mathfrak{C}(X), \mathfrak{S})$. It is easily seen that $(X, \mathfrak{S}, \pi)$ satisfies condition (C) (use Los' Theorem) and the correspondence in 1.1 reduces to the one in 3.1 of [7].

Many of the standard consequences of the Feferman-Vaught reduction can be obtained from 1.1 as well. A form of 5.1–5.3 and 5.5, 5.6 of [7] are given below. The proofs are left to the reader.

THEOREM 1.2. *If* $\mathfrak{S}_X$ *is elementarily equivalent to* $\mathfrak{S}_x'$ *for each* $x \in X$, *then* $P_\Omega(\mathfrak{C}(X), \mathfrak{S})$ *is elementarily equivalent to* $P_\Omega(\mathfrak{C}(X), \mathfrak{S}')$. *A similar statement is true when "elementarily equivalent" is replaced by "elementary subsystem".*

It is interesting to note that 1.2 shows $ThP_\Omega(\mathfrak{C}(X), \mathfrak{S})$ does not depend on the topology of $\mathfrak{S}$.

THEOREM 1.3. *If* A *and* A' *are elementarily equivalent and* $\mathfrak{C}(X)$ *and* $\mathfrak{C}(Y)$ *are elementarily equivalent, then so are* $P_\Omega(\mathfrak{C}(X), \mathfrak{A})$ *and* $P_\Omega(\mathfrak{C}(Y), \mathfrak{A}')$.

THEOREM 1.4. $Th\{P_\Omega(\mathfrak{C}(X), \mathfrak{A})\}$ *is decidable whenever* $Th\{\mathfrak{C}(X)\}$ *and* $Th\{\mathfrak{A}\}$ *are decidable.*

THEOREM 1.5. *If* $\mathcal{K}$ *is an isomorphism closed class of* L_B*-structures with* $Th\mathcal{K}$ *decidable and* $\mathfrak{M}$ *is a class of* L*-structures with* $Th\mathfrak{M}$ *decidable, then* $ThP_\Omega(\mathcal{K}, \mathfrak{M})$ *is decidable.*

The following is also of interest.

THEOREM 1.6. *If a sentence* ϕ *holds in some member of* $P_\Omega(\mathcal{K}, \mathfrak{M})$, *then* ϕ *will hold in some* $P_\Omega(\mathfrak{C}(X), \mathfrak{S})$ *where* $\mathfrak{C}(X) \in \mathcal{K}$ *and* $(X, \mathfrak{S}, \pi)$ *is a finite direct sum of constant sheaves of members in* $\mathfrak{M}$.

Proof. Suppose ϕ corresponds to a partitioning sequence $\zeta = \langle \Phi, \theta_0, \cdots, \theta_{m-1}\rangle$ by 1.1 and $P_\Omega(\mathfrak{C}(X), \mathfrak{T}) \models \phi$ for some $\mathfrak{C}(X) \in \mathcal{K}$ and a sheaf $(X, \mathfrak{T}, \pi)$ of structures in $\mathfrak{M}$. Let $t = \{j < m : K_{\theta_j}{}^{X,\mathfrak{T}} \neq 0\}$. Since ζ is a partitioning sequence for $j \in t$ there exist $\mathfrak{A}_j \in \mathfrak{M}$ such that θ_j holds in $\mathfrak{A}_j$ while θ_i fails in $\mathfrak{A}_j$ for $i \neq j$. Let $(X, \mathfrak{S}, \pi)$ denote the sheaf whose restriction, for each $j \in t$, to $X_j = K_{\theta_j}{}^{X,\mathfrak{T}}$ is a constant $\mathfrak{A}_j$-sheaf over X_j. Then $K_{\theta_i}{}^{X,\mathfrak{T}} = K_{\theta_i}{}^{X,\mathfrak{S}}$ for each $i < m$ and $P_\Omega(\mathfrak{C}(X), \mathfrak{S}) \models \phi$ follows from 1.1.

Suppose η is a standard acceptable sequence with one free variable and Ω is a set of standard acceptable sequences. The L_Ω-structure

$$\langle Q_\eta^{\mathfrak{C}(X),\mathfrak{S}}, R_\xi\rangle_{\xi\in\Omega},$$

where R_ξ is the restriction of $Q_\xi^{\mathfrak{C}(X),\mathfrak{S}}$ to $Q_\eta^{\mathfrak{C}(X),\mathfrak{S}}$, for $\xi \in \Omega$, is called a *relativized sectional structure* or the structure obtained by relativizing $P_{\Omega'}(\mathfrak{C}(X), \mathfrak{S})$ to Q_η where $\Omega' = \Omega \cup \{\eta\}$. Standard arguments (cf., [7]) show that 1.1–1.6 remain true when sectional structures are replaced by relativized sectional structures.

2. This section combines a few known sheaf-theoretic representation results with 1.5 to produce decidability results. The first applications 2.1–2.4 deal with Boolean extensions of certain finite structures $\mathfrak{A}$, i.e., members of $P_{\Omega_0}(BA, \mathfrak{A})$. 2.1–2.4 can also be obtained using generalized limit powers and reduced powers (cf., [15, 16, 17, 18]). However, these techniques do not seem to yield 2.5–2.7.

Since condition (C) holds for sheaves of models of positively model-complete theories, the following lemma permits results from section 1 to be applied to constant $\mathfrak{A}$-sheaves for $\mathfrak{A}$ finite.

LEMMA. *If $\mathfrak{A}$ is a finite L-structure, every L-formula ϕ with free variables $v_{i_1}, \cdots, v_{i_k}$ is equivalent, relative to $Th_L\{\mathfrak{A}\}$, to a positive existential formula.*

Proof. Let $L(\mathfrak{A})$ denote the language obtained by adding the elements of $\mathfrak{A}$ to L as constants. For each $a \in \mathfrak{A}^k$ let ψ_a denote the $L(\mathfrak{A})$-formula $(v_{i_1} = a_1) \wedge \cdots \wedge (v_{i_K} = a_K)$. Relative to $Th_{L(\mathfrak{A})}\{\mathfrak{A}\}$, ϕ is equivalent to the disjunction of all ψ_a's where a satisfies ϕ in $\mathfrak{A}$. A positive existential L-formula equivalent to ϕ, in $Th_L\{\mathfrak{A}\}$, is obtained by discarding the constants in favor of existential quantifiers.

Since a finite $\mathfrak{A}$ has a decidable theory and the theory of BA's is decidable (for example, see Rabin [14]), the above lemma and 1.5 yield.

THEOREM 2.1. *$P_{\Omega_0}(BA, \mathfrak{A})$ is decidable for every finite structure $\mathfrak{A}$.*

The other consequences of 1.1 apply to Boolean extensions as well. For example, if two BA's are elementarily equivalent, so are the corresponding Boolean extensions of finite structure. It also follows that every Boolean extension of a finite structure has a decidable theory. Corollaries of 2.1 are given below. Similar special cases of the results mentioned above hold but will not be stated.

A finite universal algebra, with more than one element, in which every function is a polynomial is called a *primal* algebra. It is known ([9], [10]) that the nontrivial members of the variety generated by a primal algebra A coincides, up to isomorphism, with $P_{\Omega_0}(BA, \mathfrak{A})$.

COROLLARY 2.2. *The theory of the variety generated by a primal algebra is decidable.*

If A is a finite chain with n elements considered as a distributive lattice, then, up to isomorphism, $P_{\Omega_0}(BA, \mathfrak{A})$ is the class of all Post algebras with order n (see [1], [6]).

COROLLARY 2.3. *The theory of Post algebras of order n is decidable.*

The work of McCoy and Montgomery [11] (see also [13]) shows, for each prime p, that the class of rings (with 1) that satisfy $x^p = x$ and $px = 0$ coincides (up to isomorphism) with the class of all Boolean extensions of the field $GF(p)$.

COROLLARY 2.4. *The theory of rings (with 1) that satisfy $x^p = x$ and $px = 0$ for a prime p is decidable.*

We improve 2.4 below. An m-ring ($m > 1$) is a ring with 1 that satisfies the identity $x^m = x$. A sheaf $(X, \mathfrak{S}', \pi')$ is a subsheaf of $(X, \mathfrak{S}, \pi)$ if $\mathfrak{S}'$ is open in $\mathfrak{S}$, each stalk $\mathfrak{S}_x'$ is a substructure of $\mathfrak{S}_x$, and π' equals π restricted to $\mathfrak{S}'$. A basic representation theorem for p^e-rings given in Arens, Kaplansky [2] (also see [13]) states that every countable such ring is isomorphic to $\Gamma(X, \mathfrak{S})$ for some Boolean space X and subsheaf $(X, \mathfrak{S}, \pi)$ of the constant $GF(p^e)$-sheaf over X. Such rings turn out to be relativized sectional structures.

In order to use the Arens, Kaplansky result to obtain decidability we need the following application of the relativized form of 1.5.

THEOREM 2.5. *For a finite structure A, $Th\{\Gamma(X, \mathfrak{S}) : (X, \mathfrak{S}, \pi)$ is a subsheaf of a constant $\mathfrak{A}$-sheaf over a Boolean space $X\}$ is decidable.*

Proof. Suppose $\mathfrak{A}$ is a fixed finite L-structure. Fix a one-to-one enumeration of the elements of $\mathfrak{A}$, say a_i for $i < m$, and an enumeration of the substructures of $\mathfrak{A}$, say $\mathfrak{A}_i$ for $i < k$. Let L_B denote the language for BA's with k distinguished ideals. Let K_R consist of all structures $\langle \mathfrak{B}, I_i \rangle_{i<k}$ where $\mathfrak{B}$ is a BA, each I_i is an ideal of $\mathfrak{B}$, and (i) $I_i \subseteq I_j$ whenever $\mathfrak{A}_j \subseteq \mathfrak{A}_i$, (ii) $1^{\mathfrak{B}}$ belongs to the ideal generated by $\bigcup \{I_i : i < k\}$, and (iii) $I_i \cap I_j = I_t$ whenever $\mathfrak{A}_t$ is the substructure generated by $\mathfrak{A}_i \cup \mathfrak{A}_j$. Since these axioms can be expressed in L_B, $Th\ K_R$ is a finitely axiomatizable extension of the theory of BA's with a sequence of distinguished ideals. Since the latter is decidable (by Rabin [14]), $Th\,K_R$ is decidable.

Suppose $\langle C(X), I_i \rangle_{i<k} \in K_R$ where $\langle X, U_i \rangle_{i<k}$ is a Boolean space with open subsets U_i corresponding to the ideals I_i. If $\mathfrak{S}' = U_{i<k} U_i \times A_i$ and $\pi'(x, a) = x$, it is easy to see that $(X, \mathfrak{S}', \pi')$ is a subsheaf of the $\mathfrak{A}$-constant sheaf over X. On the other hand, for any subsheaf $(X, \mathfrak{S}', \pi')$ of the $\mathfrak{A}$-constant sheaf over X, let $U_i = \{x \in X : \mathfrak{A}_i \subseteq \mathfrak{S}'_x\}$. Then U_i is an open subset of X, $\mathfrak{S}' = \bigcup_{i<k} U_i \times \mathfrak{A}_i$, and $\langle \mathfrak{C}(X), I_i \rangle_{i<k}$ belongs to K_R where I_i is the ideal corresponding to U_i. Thus, members of K_R correspond to subsheaves of constant $\mathfrak{A}$-sheaves. Moreover, suppose $(X, \mathfrak{S}, \pi)$ is a constant $\mathfrak{A}$-sheaf, $(X, \mathfrak{S}', \pi')$ a subsheaf determined by open subsets U_i, $i < k$, and $\sigma \in \Gamma(X, \mathfrak{S})$. Then $\sigma \in \Gamma(X, \mathfrak{S}')$ if and only if, for each $a_i \in \mathfrak{A}$, $K_{v_0=a_i}{}^{X,\mathfrak{S}}(\sigma) \subseteq \bigcup \{U_j : a_i \in \mathfrak{A}_j\}$. Let $\eta = \langle \Phi, \theta_0, \cdots, \theta_{m-1} \rangle$ where θ_i is $v_0 = a_i$ and Φ is the conjunction, for $i < m$, of L_B-formulas stating that X_i is contained in the ideal generated by $\bigcup \{I_j : a_i \in \mathfrak{A}_j\}$. Observe that θ_i is a formula in the language $L(\mathfrak{A})$ obtained from L by adding the elements of $\mathfrak{A}$ as constants. The above shows that $\Gamma(X, \mathfrak{S}') = Q_\eta^{\mathfrak{C}(X), \mathfrak{S}}$. Let Ω_0 be the set of standard acceptable sequences corresponding to the symbols in L and $\Omega' = \Omega_0 \cup \{\eta\}$. Hence the class of L-structures obtained by relativizing members of $P_{\Omega'}(K_R, \mathfrak{A})$ to Q_η is exactly the class of all $\Gamma(X, \mathfrak{S}')$ where $(X, \mathfrak{S}', \pi')$ is a subsheaf of a constant $\mathfrak{A}$-sheaf over a Boolean space. Since $Th\ K_R$ is decidable and $Th_{L(\mathfrak{A})}\{\mathfrak{A}\}$ is decidable (and positively model-complete), 2.5 follows from the relativized version of 1.5.

The Arens, Kaplansky result mentioned earlier yields the following corollary to 2.5.

COROLLARY 2.6. *The theory of p^e-rings is decidable.*

For a given $m > 1$ and prime power p^e, the class $R(p^e, m)$ of all p^e-rings satisfying $x^m = x$ has a decidable theory as a consequence of 2.6. Standard facts about m-rings (see [13]) show that every m-ring is a product of p^e-rings (that are also m-rings) for a finite number of prime powers p^e that can be calculated. $P_m = \{p^e : p^e - 1$ divides $m - 1\}$ is finite. Thus, the class of m-rings coincides, up to isomorphism, with the class of all finite products of rings in $\bigcup \{R(p^e, m) : p^e \in P_m\}$. The following is a consequence of these observations and the standard Feferman-Vaught results about products.

COROLLARY 2.7. *For $m > 1$, the theory of m-rings is decidable.*

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