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Monadic algebras with finite degree

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Abstract

A monadic algebra A has finite degree n if A/M has at most 2^n elements for every maximal ideal M of A and this bound is obtained for some M. Every countable monadic algebra with a finite degree is isomorphic to an algebra $\Gamma(X, S)$ where X is a Boolean space and S is a subsheaf of a constant sheaf with a finite simple stalk. This representation is used to prove that every proper equational class of monadic algebras has a decidable first-order theory.

Monadic algebras were introduced by Halmos [3]. The notion coincides with that of one dimensional cylindric algebra. Relevant definitions are given in Section 4.

Several representations of monadic algebras are known. For example, the cylindric representation theory yields representations as subdirect products of set algebras and the cylindric duality theory yields representations as sectional structures of sheaves. The main result, Theorem 8.1, establishes a new representation for countable monadic algebras with finite degree. In terms of sheaves the result stated that a certain dual sheaf is isomorphic to a subsheaf of a constant sheaf with a finite simple stalk. In non sheaf language, the result says that a countable algebra with finite degree can be represented as an algebra of special continuous functions. A detailed formulation is given in 8.3.

The importance of the type of representation developed is due to its relationship with decision problems. A method that establishes decidability using such representations is given in [2]. This technique is applied in Section 9 to show that every proper equational class of monadic algebras has a decidable theory. The decision problem for the class of all monadic algebras remains open.

We conclude in Section 10 with a remark concerning reduced products and sectional representations. Section 1–7 are preliminaries to the main result. Several general characterizations of subsheaves of constant sheaves are given in Section 2.

§1. Sheaves

The notion of sheaf is basic to this paper. The main concepts are defined below. A reader who is unfamiliar with the elementary properties of sheaves and sectional structures can consult Part 1 of R. S. Pierce's monograph [5] where the general facts are laid out for rings.

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A sheaf of algebras with type μ is a triple (X, S, π) where (i) X and S are topological spaces; (ii) π is a local homeomorphism from S onto X; (iii) for each $x \in X$, $\pi^{-1}(x) = S_x$ is the universe of an algebra with type μ; (iv) the natural partial operations induced on S by the operations on each S_x are continuous. The algebras S_x are called the stalks of the sheaf. If the meaning of X or π is clear from the context, it wil be dropped from the notation.

A section of a sheaf (X, S, π) is a continuous map $\sigma: X \to S$ such that $\pi\sigma$ is the identity on X. The subset of $\prod_{x \in X} S_x$ that consists of all sections is denoted by $\Gamma(X, S)$. Condition (iv) above implies that $\Gamma(X, S)$ is closed under all of the operations of the product $\prod_{x \in X} S_x$; so it inherits the structure of an algebra with type μ. For an algebra A, regarded as a discrete topological space, (X, S, π) is a constant A-sheaf if $S = X \times A$ with the product topology and π is the projection. The constant A-sheaf over X is denoted by $S(X, A)$.

A map $h: (X, S, \pi) \to (X, S', \pi')$ of sheaves, usually written $h: S \to S'$, is a continuous function h from S into S' such that $\pi = \pi' \circ h$ and $h_x: S_x \to S'_x$ is a homomorphism for each $x \in X$ where h_x is the restriction of h to S_x. A sheaf map $h: S \to S'$ is an isomorphism, written $h: S \cong S'$, if it is a homeomorphism of S onto S'. This implies that each h_x is an isomorphism.

If (X, S, π) and (X, S', π') are sheaves and S' is a subset of S, we say that S' is a subsheaf of S if the inclusion function is a sheaf map. It is well known that a subset S' of a sheaf S has at most one structure making it a subsheaf of S. Such a structure exist if (i) $\pi(S') = X$, (ii) S' is open in S, and (iii) for every $x \in X$, $S'_x = \pi^{-1}(x) \cap S'$ is a subsheaf of S_x.

We assume that X is a Boolean space throughout the paper.

§2. Subsheaves of constant sheaves

A standard way to construct subsheaves of $S(X, A)$ is well known. A collection $\{U_i: i \in I\}$ of open subsets of X and a family $\{A_i: i \in I\}$ of subalgebras of A are specified with the properties that $\{U_i: i \in I\}$ covers X and is closed under finite intersection and that $U_i \subseteq U_j$ implies $A_i \supseteq A_j$ for all i and j in I. The set $\bigcup_{i \in I} U_i \times A_i$, denoted $S(U_i; A_i)_{i \in I}$, inherits the structure of a subsheaf of $S(X, A)$. $S(U_i, A_i)_{I \in I}$ is called a *nice* subsheaf.

The objective of this section is to show that every subsheaf of $S(X, A)$ is nice whenever A is a finite algebra. Along the way other descriptions of subsheaves are obtained. The ideas here are closely related to those in Section 12 of [5]. We assume that A is finite.

First introduce a lower semi-continuous function $X \to S(A)$, the set of all subalgebras of A partially ordered by $\subseteq$. A function $\delta: X \to S(A)$ is *lower semi-continuous* if $\{x \in X: \delta(x) \supseteq B\}$ is open for every $B \in S(A)$.

Let π_2 denote the projection of $X \times A$ onto A.

For a subsheaf T of $S(X, A)$ define $\delta_T: X \to S(A)$ by $\delta_T(x) = \pi_2(T_x)$ for each $x \in X$.

LEMMA 2.1. (a) *δ_T is a lower semi-continuous function.*

(b) *For subsheaves S, T of $S(X, A)$, $S = T$ iff $\delta_T = \delta_S$.*

Proof. (a). Suppose $\delta_T(x) \supseteq B$. Then there exist a section $\sigma_b \in \Gamma(X, T)$ for each $b \in B$ such that $\sigma_b(x) = (x, b)$. Since T is an open subset of $X \times A$ and the topology on T is induced from $X \times A$, there is a clopen neighborhood N of x such that $\sigma_b(y) = (y, b)$ for all $y \in N$. Since A is finite there is a clopen N such that $\sigma_b(y) = (y, b)$ for all $b \in B$ and $y \in N$. Thus, N is a neighborhood of x contained in $\{z : \delta_T(z) \supseteq B\}$. (*b*). Obvious.

LEMMA 2.2. *Given a lower semi-continuous function $\delta: X \to S(A)$ define $U_B = \{x \in X : \delta(x) \supseteq B\}$ for each $B \in S(A)$. Then*

(i) *U_B is open for each $B \in S(A)$,*

(ii) *$\{U_B : B \in S(A)\}$ covers X,*

(iii) *for all $B, C \in S(A)$, $U_B \cap U_C = U_D$ where D is the subalgebra of A generated by B and C.*

The proof is obvious.

Observe, as a consequence of (iii), that $B \subseteq C$ implies $U_B \supseteq U_C$. Also, in case A has a smallest subalgebra M, condition (ii) can be replaced by the condition $U_M = X$.

LEMMA 2.3. *Given $\{U_B : B \in S(A)\}$ with properties* 2.2(i)–2.2(iii).

(1) *There exist a subset I of $S(A)$ such that $\{U_B : B \in I\}$ covers X, is closed under intersection, and, for all $B, C \in I$, $U_B \subseteq U_C$ implies $C \subseteq B$.*

(2) *The nice subsheaf $S = S(U_B, B)_{B \in I} = \bigcup \{U_B \times B : B \in S(A)\}$.*

Proof. (1). For $B, C \in S(A)$ define $B \sim C$ iff $U_B = U_C$. Clearly $\sim$ is an equivalence relation on $S(A)$. Moreover, if $B \sim C$ and D is the subalgebra generated by B and C, then, by 2.2(iii), $U_D = U_B \cap U_C = U_B$, i.e., $D \sim B$. Since A is finite each equivalence class of $\sim$ contains a largest member under $\subseteq$. Let I be the subset of $S(A)$ that consists of the $\subseteq$-largest member of each $\sim$ equivalence class. The following properties are now clear:

(a) if $B \in S(A)$, there exist $C \in I$ such that $U_B = U_C$,

(b) if $B \in S(A)$ and $C \in I$ such that $U_B = U_C$, then $B \subseteq C$.

From (a) and 2.2(ii) it follows that $\{U_B : B \in I\}$ covers X and from (a) and 2.2(ii) it follows that $\{U_B : B \in I\}$ is closed under intersection. Now, suppose $B, C \in I$ and $U_B \subseteq U_C$. By 2.2(iii), $U_B = U_B \cap U_C = U_D$ where D is the subalgebra generated by B and C. By (b), $B \supseteq D \supseteq C$ as desired. Hence (1) holds.

(2) By (1), $S=S(U_B, B)_{B\in I}$ is a nice subsheaf of $S(X, A)$ and is clearly contained in $\bigcup\{U_B\times B\colon B\in S(A)\}$. On the other hand, for $B\in S(A)$, by (a) there is a unique $C\in I$ such that $U_B=U_C$ and, by (b), $B\subseteq C$. Hence

$$U_B\times B=U_C\times B\subseteq U_C\times C\subseteq S.$$

Thus, $S(U_B, B)_{B\in I}=\bigcup\{U_C\times C\colon C\in S(A)\}$ as desired.

Remark. For the nice subsheaf S in 2.3(2), δ_S can also be described in terms of the family $\{U_B\colon B\in S(A)\}$. Namely, for $x\in X$, $\delta_S(x)$ is the subalgebra of A generated by all B's with $x\in U_B$.

THEOREM 2.4. *Every subsheaf of* $S(X, A)$ *is nice. In addition, subsheaves correspond in a unique way with lower semi-continuous functions* $\delta\colon X\to S(A)$ *and also with families* $\{U_B\colon B\in S(A)\}$ *of subsets of X that satisfy* 2.2(i)–2.2(iii).

Proof. Let T be a subsheaf of $S(X, A)$. By 2.1 $\delta=\delta_T$ may be used in 2.2 to produce a family of sets $\{U_B\colon B\in S(A)\}$ with properties 2.2(i)–2.2(iii). This family yields, by 2.3, a nice subsheaf $S=\bigcup\{U_B\times B\colon B\in S(A)\}\subseteq T$. Conversely, if $(x, s)\in T$, $s\in\pi_2(T_x)$ and $x\in\{y\colon\pi_2(T_y)\supseteq\pi_2(T_x)\}=U_{\pi_2(T_x)}$. Thus, $(x, s)\in U_{\pi_2(T_x)}\times\pi_2(T_x)\subseteq S$. Hence $T=S$ is nice as desired.

Suppose that a lower semi-continuous function δ is given and that S is the nice subsheaf produced using 2.2 and 2.3. Since $x\in U_{\delta(x)}=\{y\colon\delta(y)\supseteq\delta(x)\}$, $\{x\}\times\delta(x)\subseteq U_{\delta(x)}\times\delta(x)\subseteq S$; so $\delta(x)\subseteq\delta_S(x)$. On the other hand, $x\in U_{\delta_S(x)}$ implies $\delta(x)\supseteq\delta_S(x)$. Hence $\delta=\delta_S$.

Now, suppose we start with a family of sets $\{V_B\colon B\in S(A)\}$ satisfying 2.2(i)–(iii) and construct in turn S, δ_S and finally a family of sets $\{U_B\colon B\in S(A)\}$ using δ_S in 2.2. It is easy to check that $x\in U_B$ iff $\delta_S(x)\supseteq B$ iff $x\in V_B$ using the remark prior to 2.4. Thus $U_B=V_B$ as desired.

By the previous results a subsheaf S of $S(X, A)$ has the form $S=S(U_B, B)_{B\in I}$ where $I\subseteq S(A)$. (Cf., 2.3.) Call the set of pairs (U_B, B) where $B\in I$ the *standard components* of S. This description has the advantage that some superfluous information has been removed.

It is well-known that $\Gamma(X, S(X, A))$ is isomorphic to the algebra of all continuous functions from X into A. The next result relates $\Gamma(X, S)$, where S is a subsheaf of a constant sheaf, with algebras of special continuous functions from X into A.

Suppose that a subsheaf S of $S(X, A)$ has the form $S=S(U_B, B)_{B\in I}$ given by 2.3. For each $B\in S(A)$ let $X_B=\bigcap\{X-U_C\colon C-B\neq 0\}$. Each X_B is a closed subset of X.

THEOREM 2.5. *The map that sends* σ *to* $\pi_2\sigma$ *is an isomorphism of* $\Gamma(X, S)$ *onto the algebra K of all continuous functions from X into A that take values in B on the closed set* X_B *for every subalgebra B of A.*

Proof. Suppose $\sigma\in\Gamma(X, S)$ and $x\in X_C$. Then $\pi_2\sigma(x)\in C$; for if not, $\delta_S(x)-C\neq 0$ so $x\in X_C\subseteq X-U_{\delta_S(x)}$ which contradicts $x\in U_{\delta_S(x)}$. Hence $\pi_2\delta\in K$ whenever $\sigma\in\Gamma(X, S)$. It remains to show, for $f\in K$ and σ the unique section in $\Gamma(X, S(X, A))$ such that $\pi_2\sigma=f$, that $\sigma\in\Gamma(X, S)$. So for $x\in X$ we need to show $\pi_2\sigma(x)=f(x)\in\delta_S(x)$. Since $f\in K$ it is enough to show $x\in X_{\delta_S(x)}$.

Since $x\in U_{\delta_S(x)}$, for any B, $x\in U_B$ iff $x\in U_B\cap U_{\delta_S(x)}=U_C$ where C is the subalgebra generated by B and $\delta_S(x)$. Hence it follows that $x\in X_{\delta_S(x)}$ since $x\notin U_B$ for every $B\not\supseteq\delta_S(x)$.

§3. A-coordinates

A sheaf (X, S) *admits A-coordinates* if it is isomorphic to a subsheaf of $S(X, A)$. If (X, S) is isomorphic to $S(X, A)$, we say that (X, S) admits *full* A-coordinates. (X, S) admits *local* A-coordinates if every $x\in X$ has a neighborhood U such that $\pi^{-1}(U)$ admits A-coordinates. If the sheaf under consideration is clear from the context we abuse the above terminology by saying that an open set U admits A-coordinates in place of saying $\pi^{-1}(U)$ admits A-coordinates.

The following lemmas provide methods for producing A-coordinates. The first is a direct application of the partition property (cf. [5]).

LEMMA 3.1. *If a sheaf over a Boolean space admits local A-coordinates, it admits A-coordinates. If it admits local full A-coordinates, it admits full A-coordinates.*

LEMMA 3.2. *If a sheaf admits A-coordinates, it admits A^*-coordinates for every A^* in which A can be embedded.*

Lemma 3.1 can be weakened slightly for certain space. Call a Boolean space *partitionable* if every open set is a union of a famliy of pairwise disjoint clopen sets. A sheaf is *partitionable* if its base is.

LEMMA 3.3. *If an open set U of a partitionable Boolean space admits local (full) A-coordinates, then U admits (full) A-coordinates.*

Proof. Use 3.1 on each clopen set and then piece the isomorphisms together.

The following lemma is used to extend coordinates.

LEMMA 3.4. *Suppose S is a sheaf over X, U is open in X, $\bar{S}$ is a subsheaf of $S(U, A^*)$, and $\bar{h}:\bar{S}\cong\pi^{-1}(U)$. Furthermore, suppose there exist $\sigma_a\in\Gamma(X, S)$ for every $a\in A$ and a monomorphism $f:A\to A^*$ such that*

(a) *for all $y\in U$, $f(A)$ is a subalgebra of $\pi_2(\bar{S}_y)$,*

(b) *the map sending a to $\sigma_a(y)$ is a monomorphism of A into S_y for all $y\in X$ and an isomorphism for all $y\in X-U$,*

(c) *for each* $y \in U$ *and* $a \in A$, $\bar{h}(y, f(a)) = \sigma_a(y)$. *Then*

(i) $S' = \bar{S} \cup (X \times f(A))$ *is a subsheaf of* $S(X, A^*)$, *and*

(ii) *the map* h' *defined for* $(x, a) \in S'$ *by*

$$h'(x, a) = \begin{cases} \bar{h}(x, a) & \text{if } x \in U \\ \sigma_{f^{-1}(a)}(x) & \text{if } x \in X - U \end{cases}$$

is an isomorphism $S' \cong S$.

Proof. (i) S' projects onto X, each stalk S'_y is a subalgebra of $\{y\} \times A^*$, and S' is an open subset of $X \times A^*$.

(ii) To see that $h': S' \cong S$ is an isomorphism, observe it is one-one, and

(d) $$h'(y, f(b)) = \sigma_b(y) \quad \text{for} \quad y \in X, \ b \in A.$$

$h'(y, f(b)) = \sigma_b(y)$ for $y \in X - U$ by the definition of h'. For $y \in U$, the definition of h' and (c) give $h'(y, f(b)) = \bar{h}(y, f(b)) = \sigma_b(y)$ as desired.

Since (d) implies that h' is onto and that $h'(N \times \{f(b)\}) = \sigma_b(N)$ for every clopen subset N of X and $b \in A$, h' is a homeomorphism onto S. The fact it is an isomorphism now follows from (b).

§4. Monadic algebras

This section contains general information about monadic algebras.

A *monadic algebra* is a structure $\langle A, +, \cdot, -, 0, 1, c \rangle$ where $\langle A, +, \cdot, -, 0, 1 \rangle$ is a Boolean algebra (BA) and c is a quantifier on this BA, i.e., $c0 = 0$, $x \leqslant cx$, and $c(x \cdot cy) = cx \cdot cy$. x is *closed* if $cx = x$. Denote the class of all monadic algebras by CA_1.

A simple CA_1 is a non-trivial BA with a quantifier c such that $cx = 1$ if $x \neq 0$. For each $m = 1, 2, \ldots$ let A_m be the simple CA_1 whose Boolean part is the BA of all subsets of $\{1, \ldots, m\}$. Let A_0 denote a one element CA_1. For each $m < \omega$ let V_m denote the variety generated by A_m. In [4] Monk showed that the varieties of CA_1's form an $\omega + 1$ chain $V_0 < V_1 < V_2 < \cdots < CA_1 = V_\infty$. We say that A has degree $\leqslant m$ if $A \in V_m$. A has degree m if it has degree $\leqslant m$ but does not have degree $\leqslant k$ for any $k < m$. For finite m this agrees with the definition given in the introduction.

As a special case of the duality theory [1] every nontrivial CA_1 is isomorphic to $\Gamma(X, S)$ where (X, S) is a sheaf of simple CA_1's over a Boolean space. In this representation, for $m < \omega$, $\Gamma(X, S)$ has degree $\leqslant m$ if and only if S_x is embeddable in A_m, for every $x \in X$. For a sheaf (X, S) of simple CA_1's define $\delta_S^*: X \to \omega \cup \{\infty\}$ by setting $\delta_S^*(x) = m$ if $S_x \cong A_m$ and $\delta_S^*(x) = \infty$ if S_x is infinite. Call δ_S^* the *degree function* of S. We say that (X, S) or S is an *m-sheaf* if $\delta_S^*(x) \leq m$ for all $x \in X$. Thus, S is an m-sheaf if and only if it is the dual sheaf of a CA_1 with degree $\leq m$.

For each $m < \omega$ let $U_m = \{x \in X : \delta_S^*(x) \geq m\}$.

LEMMA 4.1. *U_m is open for each $m < \omega$.*

Proof. For each $x \in U_m$ there exist 2^m sections $\sigma_i \in \Gamma(X, S)$ $(i < 2^m)$ such that $\sigma_i(x) \neq \sigma_j(x)$ for all $i \neq j$. Thus, $\prod_{i<j<2^m} c(\sigma_i \oplus \sigma_j)(x) = 1_x$ where $\oplus$ denotes symmetric difference. Hence, $\prod_{i<j} c(\sigma_i \oplus \sigma_j)(y) = 1_y$ for all y in some clopen neighborhood N of x. Since S_y contains at least 2^m distinct elements for each $y \in N$, $N \subseteq U_m$.

The following two lemmas will be used to introduce coordinates.

LEMMA 4.2. *For every $x \in U_m$ there exist a clopen neighborhood $N \subseteq U_m$ of x and $\sigma_a \in \Gamma(N, S)$ for $a \in A_m$ such that the map $A_m \to S_y$ that assigns a to $\sigma_a(y)$ is a monomorphism for all $y \in N$.*

Proof. A_m is isomorphic to a subalgebra of S_x since $x \in U_m$. Say there exist $\sigma'_a \in \Gamma(X, S)$ for $a \in A_m$ such that the isomorphism sends a to $\sigma'_a(x)$. The isomorphism property can be expressed by a finite number of conditions:

$$\begin{array}{ll} \prod_{a \neq b} c(\sigma'_a \oplus \sigma'_b)(x) = 1_x & \\ c\sigma'_a(x) = 1_x & \text{for each } a \neq 0 \\ (\sigma'_a + \sigma'_b)(x) = \sigma'_{a+b}(x) & \text{for all } a, b \in A \\ -\sigma'_a(x) = \sigma'_{-a}(x) & \text{for each } a \in A \\ \sigma'_1(x) = 1_x & \\ \sigma'_0(x) = 0_x & \end{array}$$

Thus there exists a clopen neighborhood N of x such that the above set of conditions holds for all $y \in N$. Hence, for each $y \in N$, the map that sends a to $\sigma'_a(y)$ gives a monomorphism of $A_m \to S_y$. Choose $\sigma_a = \sigma'_a \mid N$.

LEMMA 4.3. *For $x \in U_m$ choose N and σ_a's as in 4.2. Define $h: N \times A_m \to \pi^{-1}(S)$ by $h(y, a) = \sigma_a(y)$. Then h is a sheaf isomorphism of $N \times A_m$ onto its image $h(N \times A_m)$. If the map sending a to $\sigma_a(y)$ is onto for each $y \in N$, then $h: N \times A_m \cong \pi^{-1}(S)$.*

Proof. If N' is a clopen subset of N, $h(N' \times \{a\}) = \sigma_a(N')$. Thus, h establishes a one-one correspondence between neighborhood bases of $N \times A_m$ and $h(N \times A_m)$.

The representation result 8.1 will apply to countable monadic algebras. If A is a countable CA_1, its closed elements form a countable BA and hence the dual sheaf (X, S) of simple CA_1's such that $A \cong \Gamma(X, S)$ has the property that X is separable. In particular, every open set in X is a union of a countable family of pairwise disjoint clopen sets; i.e., the dual sheaf of a countable CA_1 is partitionable.

§5. The type of a point

To facilitate the induction proof in section 8 we introduce the following notion. Suppose $X=U_1\supseteq U_2\supseteq\cdots\supseteq U_n\neq 0$ is a decreasing sequence of open sets. A point $x\in X$ has *type m* (relative to this fixed sequence) where $1\leqslant m<n$ if $x\in U_m\cap(\bar{U}_n-U_{m+1})$.

The main facts are

PROPOSITION 5.1. (a) *If x has type m, then $x\in\bar{U}_{m+1}-U_{m+1}$.*

(b) *Every $x\in\bar{U}_n-U_n$ has a unique type.*

Proof. (a) $U_{m+1}\supseteq U_n$ and $x\in\bar{U}_n$ so $x\in\bar{U}_{m+1}$. But $x\notin U_{m+1}$.

(b) $x\in U_1\cap(\bar{U}_n-U_n)$. Let $m=\max\{i:x\in U_i\}$. Then $x\in U_m-U_{m+1}$. Hence x has type m. It is clear that a point cannot have two distinct types.

§6. Partial coordinates

The main results of the paper deal with coordinations using finite simple monadic algebras. It was shown in section 2 that a subsheaf S' of a constant A-sheaf, A finite, is nice. This allows us to examine A-coordinates in terms of the standard components of S'.

Suppose (X, S) is a sheaf, U is an open subset of X, and A-coordinates of $\pi^{-1}(U)$ are given by the isomorphism $h:S'\cong\pi^{-1}(U)$ where S' is a subsheaf of $S(U, A)$. Suppose the standard components of S' are the pairs (U_B, B) for $B\in I$ where $I\subseteq S(A)$. For each $B\in I$ and $b\in B$ there is $c_b^B\in\Gamma(U_B, S')$ given by $c_b^B(y)=(y, b)$. Denote $\tau_b^B=hc_b^B\in\Gamma(U_B, \pi^{-1}(U_B))$. A homomorphism $\tau^B:B\to\Gamma(U_B, \pi^{-1}(U_B))$ is defined by $\tau^B(b)=\tau_b^B$. We call τ^B a *partial A-coordinate*.

The family $\{\tau^B: B\in I\}$ of partial coordinates has the following properties.

THEOREM 6.1. (1) *For each $y\in U_B$ the composition of τ^B with the homomorphism that evaluates sections at y embeds B into the stalk of $\pi^{-1}(U)$ at y.*

(2) *For every component where $(x, a)\in U_B\times B$ with $B\in I$,*

$$h(x, a)=\tau_a^B(x).$$

Proof. (1) The map that sends b to $\tau_b^B(y)$ is a composite of the monomorphism $B\to S'_y$ that sends b to (y, b) and the monomorphism $h_y:S'_y\to\pi^{-1}(U)_y$.

(2) For $B\in I$ with $a\in B$ and $x\in U_B$, $\tau_a^B(x)=hc_b^B(x)=h(x, a)$ as desired.

§7. Combinatorial properties

Recall that A_m is the specific simple CA_1 whose underlying BA is $P(m)$, the set of all subsets of $\{1,\ldots, m\}$. The lemma below establishes a recursive amalgamation property for the class of finite simple CA_1's. Iterating the function given in 7.1 produces a function n^* that will be useful later.

LEMMA 7.1. *Suppose* $1 \leqslant k < n$. *There is an* r *with the property: for every family* $\{f_i\}_{i\in I}$ *of monomorphisms* $f_i: A_k \to A_n$, *there exist a family* $\{g_i\}_{i\in I}$ *of monomorphisms* $g_i: A_n \to A_r$ *such that* $g_i f_i = g_j f_j$ *for all* $i, j \in I$. *In fact* $r(k, n) = \sum_{i=1}^{k} \max\{\mu(i): \mu \in {}^{\{1,\ldots,k\}}\{1,\ldots,n\}$ and $\sum_{j=1}^{k} \mu(j) = n\}$ *is the least* r *with the above property and* $r(k, n) = k(n-k+1)$.

Proof. Since $n = \mu(1) + \cdots + \mu(k)$ and $\mu(i) \geqslant 1$ for each i, $\max\{\mu(i): \mu \in {}^{\{1,\ldots,k\}}\{1,\ldots,n\}$ and $\sum_j \mu(j) = n\} = n-k+1$. Hence, $r(k, n) = k(n-k+1)$.

Choose $r = r(k, n)$ and suppose $\{f_i\}_{i\in I}$ is a family of monomorphisms $A_k \to A_n$. For each $i \in I$ define $\mu_i: \{1,\ldots,k\} \to \{1,\ldots,n\}$ by $\mu_i(j) = |f_i(\{j\})|$ for $j = 1,\ldots,k$. Since $n = \sum_j \mu_i(j)$ for each i, $r \geqslant r^* = \sum_j \max\{\mu_i(j): i \in I\}$.

Embeddings $g_i': A_n \to A_{r^*}$ are introduced in the following way. Partition $\{1,\ldots,r^*\}$ into k disjoint sets $a_1,\ldots,a_k$ where $|a_j| = \max\{\mu_i(j): i \in I\}$ for each j. For each $i \in I$, g_i' is specified by giving its values of the atoms of $P(n)$. For each $j = 1,\ldots,k$ either $\mu_i(j) = |a_j|$ or $\mu_i(j) < |a_j|$. In the first case let g_i' map the atoms contained in $f_i(\{j\})$ onto the atoms contained in a_j in any one-one way. In the second case, partition the atoms contained in a_j into $\mu_i(j)$ disjoint sets $b_1,\ldots,b_{\mu_i(j)}$ and assign the atoms contained in $f_i(\{j\})$ to the b's in a one-one way.

Observe that g_i' is defined in such a way that $g_i' f_i$ is the unique embedding $A_k \to A_{r^*}$ that sends $\{j\}$ to a_j for each atom $\{j\}$ in A_k. Hence, $g_i' f_i = g_j' f_j$ for all $i, j \in I$.

Finally, since $r^* \leqslant r$ we can choose a fixed embedding $h: A_{r^*} \to A_r$ and set $g_i = hg_j'$ for each $i \in I$. $\{g_i\}_{i\in I}$ is the desired family of embeddings.

It is easy to construct an example that shows $r(k, n)$ is the least value of r that works.

Let $N = \omega \sim 1$ and let $r(k, n)$ be any primitive recursive function $\omega \times \omega \to N$ that agrees with the function r in 7.1 for $1 \leqslant k < n$. For example $r(k, n) = \max\{1, k(n-k+1)\}$ will do. Define $\alpha: N \times N \times N \to N$ by

$$\alpha(m, n, 1) = \max\{m, r(n-1, n)\}$$
$$\alpha(m, n, k+1) = r(n-(k+1), \alpha(m, n, k))$$

Define $n^*: \omega \to \omega$ by the conditions: $0^* = 0$, $1^* = 1$, and $(n+1)^* = \alpha(n^*, n+1, n)$.

It is clear that both α and n^* are primitive recursive. The first few values of n^* are $1^* = 1$, $2^* = 2$, $3^* = 4$, and $4^* = 10$. For $1 \leqslant k < n$ also let $n_k = \alpha((n-1)^*, n, k)$.

The following properties about n^*, n_k, and r are used later.

LEMMA 7.2. (i) $r(k, n) \geqslant n$ *for* $1 \leqslant k \leqslant n$.

(ii) $\alpha(m, n, k+1) \geqslant \alpha(m, n, k)$ *for* $1 \leqslant k \leqslant n-2$.

(iii) $\alpha(m, n, k) \geqslant n$ *for* $1 \leqslant k \leqslant n-1$.

(iv) $\alpha(m, n, k) \geqslant m$ *for* $1 \leqslant k \leqslant n-1$.
(v) $n^* = n_{n-1}$.
(vi) $n^* \geqslant (n-1)^*$.
(vii) $n^* \geqslant n$.
(viii) $n_{k+1} \geqslant n_k$ *for* $1 \leqslant k \leqslant n-2$.
(ix) $r(n-k, n_{k-1}) = n_k$ *for* $1 < k \leqslant n-1$.

Proof. (i) $r(x, n) = x(n-x+1)$ on $[1, n]$ takes its minimum value n at $x=1$ and $x=n$. (ii) For $1 \leqslant k \leqslant n-2$, $1 \leqslant n \dot{-} (k+1) \leqslant n \leqslant r(n-1, n) \leqslant \alpha(m, n, 1)$; hence (ii) follows by induction on k using (i). (iii) $\alpha(m, n, k) \geqslant \alpha(m, n, 1) \geqslant n$ using (ii). (iv) is similar. (v) is obvious, (vi) and (vii) follow from (iii), (iv), (v), and (viii) follows from (ii). (ix) is immediate from the definition of n_{k-1}.

§8. Coordination theorem

The main representation result can now be given. n^* denotes the primitive recursive function introduced in section 7.

THEOREM 8.1. *Every partitionable n-sheaf admits A_{n^*}-coordinates.*

Proof. We proceed by induction on n. The case $n=0$ is trivial. For a 1-sheaf (X, S), $U_1 = X$ and $U_2 = \emptyset$. Lemma 4.3 implies that every point has a neighborhood that admits A_1-coordinates. By 3.1 S admits a full A_1-coordination. Basically this is just the Stone representation Theorem.

Assume $n \geqslant 2$ and let (X, S) be an n-sheaf. By 3.1 and 3.2 it suffices to show that each $x \in X$ has a neighborhood that admits A_m-coordinates for some $m \leqslant n^*$. If $x \in U_n$, Lemmas 4.2 and 4.3 give an isomorphism $S(N, A_n) \cong h(N \times A_n)$ for some neighborhood N of x with $N \subseteq U_n$. Since no stalk of S has dimension $> n$, $h(N \times A_n) = \pi^{-1}(N)$ so N admits A_n-coordinates. If $x \in X - \bar{U}_n$, there exist a clopen neighborhood N of x such that $\pi^{-1}(N)$ is a partitionable $(n-1)$-sheaf. By the induction hypothesis, $\pi^{-1}(N)$ admits $A_{(n-1)^*}$-coordinates. Now consider $x \in \bar{U}_n - U_n$. By 5.1(b) each point in $\bar{U}_n - U_n$ has a unique type m between 1 and $n-1$. By induction on $k = 1, \ldots, n-1$ we verify that

(*) $\begin{cases} \text{every point } x \in \bar{U}_n - U_n \text{ with type } m = n-k \text{ has a} \\ \text{clopen neighborhood that admits } A_{n_k}\text{-coordinates.} \end{cases}$

By 7.2, n_k is a nondecreasing function of k, $n^* = n_{n-1}$, $n^* \geqslant (n-1)^*$, and $n^* \geqslant n$. In view of the discussion above and Lemma 3.2, every point has a neighborhood that admits A_{n^*}-coordinates as desired.

It remains to establish (*) by induction on k.

In the case $k=1$ consider x with type $m = n-1$. By Lemma 4.2 there exist a clopen neighborhood $N' \subseteq U_{n-1}$, $x \in N'$, and $\sigma_a \in \Gamma(N', S)$ for all $a \in A_{n-1}$ such that, for each

$y \in N'$, the map that sends a to $\sigma_a(y)$ embeds A_{n-1} into S_y. Let $U_n' = N' \cap U_n$. U_n' is open in X. Since $\pi^{-1}(U_n')$ is an n-sheaf, Lemmas 4.2 and 4.3 show that it admits local full A_n-coordinates. By 3.3 $\pi^{-1}(U_n')$ admits full A_n-coordinates, i.e., there are $\tau_a \in \Gamma(U_n', S)$ for $a \in A_n$ such that the map that sends (y, a) to $\tau_a(y)$ is an isomorphism $S(U_n', A_n) \cong \pi^{-1}(U_n')$.

For each $y \in U_n'$, $a \in A_{n-1}$ $\sigma_a(y) = \tau_b(y)$ for some $b \in A_n$. Thus, there exist a monomorphism $f_y: A_{n-1} \to A_n$ such that, for $a \in A_{n-1}$,

$$\sigma_a(y) = \tau_{f_y(a)}(y).$$

Since $\{z: \sigma(z) = \tau(z)\}$ is clopen for any two sections σ, τ and A_{n-1} is finite, there is a clopen neighborhood N'' of y such that

$$\sigma_a(z) = \tau_{f_y(a)}(z)$$

for all $a \in A_{n-1}$ and $z \in N''$.

Now, for the embedding $f: A_{n-1} \to A_n$, let

$$K_f = \{z \in U_n': \sigma_a(z) = \tau_{f(a)}(z) \quad \text{for all } a \in A_{n-1}\}.$$

By the above discussion there exist a collection $\{f_i\}_{i \in I}$ of embeddings $f_i: A_{n-1} \to A_n$ such that

(i) each K_{f_i} is a nonempty open subset of U_n', and

(ii) $\{K_{f_i}: i \in I\}$ partitions U_n'.

By the combinatorial Lemma 7.1, the definition of n_1, and 3.2, there exist $\{g_i\}_{i \in I}$ where $g_i: A_n \to A_{n_1}$ such that $g_i f_i = g_j f_j$ for all $i, j \in I$. We use these functions to construct a subsheaf of $S(N', A_{n_1})$ isomorphic to $\pi^{-1}(N'')$ as an application of Lemma 3.4 with $U = U_n'$ and $X = N'$. Since $\{K_{f_i}: i \in I\}$ is an open partition of U_n', $\bar{S} = \bigcup_{i \in I} K_{f_i} \times g_i(A_n)$ is a subsheaf of $S(U_n', A_{n_1})$. $K_{f_i} \times A_n = S(K_{f_i}, A_n) \cong \pi^{-1}(K_{f_i})$ by the map that sends (y, a) to $\tau_a(y)$; thus, the map that sends (y, a) to $\tau_{g_i^{-1}(a)}(y)$ is an isomorphism $K_{f_i} \times g_i(A_n) \cong \pi^{-1}(K_{f_i})$. The map obtained by joining the pieces for each $i \in I$ yields an isomorphism $\bar{h}: \bar{S} \cong \pi^{-1}(U_n')$ where $\bar{h}(y, a) = \tau_{g_i^{-1}(a)}(y)$ if $y \in K_{f_i}$. Choose $0^* \in I$. By the choice of $\{g_i\}_{i \in I}$, $g_{0*} f_{0*} = g_i f_i$ is a monomorphism $A_{n-1} \to A_{n_1}$ such that $g_{0*} f_{0*}(A_{n-1})$ is a subalgebra of $g_i(A_n)$ for each $i \in I$. To apply 3.4 it remains to observe that, for $y \in U_n'$ and $b \in A_{n-1}$,

$$\bar{h}(y, g_{0*} f_{0*}(b)) = \sigma_b(y).$$

In fact, if $y \in K_{f_i}$, then $\bar{h}(y, g_{0*} f_{0*}(b)) = \bar{h}(y, g_i f_i(b)) = \tau_{f_i(b)}(y) = \sigma_b(y)$ by the choice of $\{g_i\}_{i \in I}$ and the definition of K_{f_i}. It follows from Lemma 3.4 that $S' = S \cup (N' \times g_{0*} f_{0*}(A_{n-1}))$ is isomorphic to $\pi^{-1}(N')$. Thus, a point $x \in \bar{U}_n - U_n$ with type $n-1$ has a neighborhood N where $\pi^{-1}(N)$ admits A_{n_1}-coordinates. This proves $(*)$ for $k = 1$.

Now suppose $1<k\leqslant n-1$ and $(*)$ is true for all $k'<k$. Suppose $x\in\bar{U}_n-U_n$ has type $m=n-k$. Thus, $x\in U_m\cap(\bar{U}_n-U_{m+1})$ and $x\in\bar{U}_{m+1}-U_m$ by 5.1.

Use Lemma 4.2 and the fact U_m is open to choose a clopen neighborhood N' of x, $N'\subseteq U_m$ and sections $\sigma_a\in\Gamma(N', S)$ for $a\in A_m$ such that the map $A_m\to S_y$ sending a to $\sigma_a(y)$ is an embedding for all $y\in N'$.

Set $U'_{m+1}=U_{m+1}\cap N'$. U'_{m+1} is an open subset of X and $\pi^{-1}(U'_{m+1})$ is an n-sheaf. For every $y\in U'_{m+1}$ either (a) $y\in U_n$ or (b) $y\in U_{m+1}-\bar{U}_n$ or (c) $y\in\bar{U}_n-U_n$ and has type m' for some $m'>m$ (and of course $m'<n$).

If (a) holds, since $\pi^{-1}(U'_{m+1})$ is an n-sheaf Lemma 4.3 implies there is a clopen neighborhood N of y, $N\subseteq U'_{m+1}\cap U_n$ such that $\pi^{-1}(N)$ admits A_n-coordinates.

If (b) holds, there exist a clopen neighborhood N of y disjoint from $\bar{U}_n$ and hence $\pi^{-1}(N)$ is an $(n-1)$ sheaf. By the induction hypothesis of the theorem, $\pi^{-1}(N)$ admits $A_{(n-1)^*}$-coordinates.

Suppose (c) holds and let $k'=n-m'$ where y has type m'. Since $m'>m$, $k'<k$. By the induction hypothesis (for k), $(*)$ holds for k' and thus y has a clopen neighborhood N such that $\pi^{-1}(N)$ admits $A_{n_{k'}}$-coordinates.

By 7.2, n_i is a nondecreasing function of i, $n_1\geqslant n$, and $n_1\geqslant(n-1)^*$. Since $k'\leqslant k-1$, Lemma 3.2 implies that $\pi^{-1}(U'_{m+1})$ admits a local $A_{n_{k-1}}$-coordination. X is partitionable so, by Lemma 3.3, $\pi^{-1}(U'_{m+1})$ admits $A_{n_{k-1}}$-coordinates. Thus $\pi^{-1}(U'_{m+1})$ is isomorphic to a subsheaf S' of $S(U'_{m+1}, A_{n_{k-1}})$.

Suppose the standard components of S' are the pairs (U_B, B) for $B\in I$ where $I\subseteq S(A_{n_{k-1}})$. By the development in Section 6, the isomorphism $S'\cong\pi^{-1}(U'_{m+1})$ yields a family $\{\tau^B\colon B\in I\}$ of partial $A_{n_{k-1}}$-coordinates.

For $B\in I$ and a monomorphism $f\colon A_m\to B$ let

$$K_f^B=\{z\in U_B\colon \text{ for every } a\in A_m\ \ \sigma_a(z)=\tau^B_{f(a)}(z)\}.$$

Since A_m is finite, K_f^B is open.

OBSERVATION 1. Every element in U'_{m+1} belongs to K_f^B for some $B\in I$ and some embedding $f\colon A_m\to B$.

For $y\in U'_{m+1}$, $S'_y=\{y\}\times B$ for some $B\in I$. Then $y\in U_B$ and, by 6.1. (1), the map that sends b to $\tau^B_b(y)$ is an isomorphism of B onto S_y. By choice of the σ_a's the map that sends a to $\sigma_a(y)$ is an embedding $A_m\to S_y$. Thus, there exist an embedding $f_y\colon A_m\to B$ such that $\sigma_a(y)=\tau^B_{f_y(a)}(y)$ for all $a\in A_m$.

OBSERVATION 2. $K_f^B\cap K_{f'}^C=\emptyset$ whenever $f\colon A_m\to B, f'\colon A_m\to C$ and $f\neq f'$.

Suppose $f(a)\neq f'(a)$ for some $a\in A_m$ and $y\in K_f^B\cap K_{f'}^C$. Then $h(y, f(a))=\tau^B_{f(a)}(y)$ $=\sigma_a(y)=\tau^C_{f'(a)}(y)=h(y, f'(a))$ by 6.1. (2). A contradiction to $f(a)\neq f'(a)$ is obtained from the fact h is one-one.

Set $\Sigma=\{f:A_m\to A_{n_{k-1}}: f \text{ is one-one}, f(A_m)\subseteq B \text{ for some } B\in I\}$. For $f\in\Sigma$ set

$$K_f=\bigcup\{K_f^B: B\in\Sigma, f(A_m)\subseteq B\}.$$

By observation 1 and 2 $\{K_f: f\in\Sigma\}$ is an open partition of U'_{m+1}. Say, $\{f\in\Sigma: K_f\neq\emptyset\}=\{f_i\}_{i\in J}$. By the combinatorial Lemma 7.1, there exist a family $\{g_i\}_{i\in J}$ of embeddings $g_i: A_{n_{k-1}}\to A_{n_k}$ (see 7.2(ix)) such that $g_i f_i=g_j f_j$ for all $i, j\in J$.

Now, for each $i\in J$, the $A_{n_{k-1}}$-coordination $S'\cong\pi^{-1}(U'_{m+1})$ of $\pi^{-1}(U'_{m+1})$ induces an $A_{n_{k-1}}$-coordination of $\pi^{-1}(K_{f_i})$ in the following way. Let $U_B^i=K_{f_i}\cap U_B$ for each $B\in I$. Then $K_{f_i}=\bigcup_{B\in I} U_B^i$. Let $S^i=\bigcup_{B\in I} U_B^i\times B$. Since S^i is the restriction of S' to K_{f_i}, the isomorphism $h: S'\cong\pi^{-1}(U'_{m+1})$ restricts to give $S^i\cong\pi^{-1}(K_{f_i})$ for each $i\in J$.

Lemma 3.4 with $U=U'_{m+1}$ and $X=N'$ will be applied to construct a A_{n_k}-coordination for $\pi^{-1}(N')$. First $\bar{S}$ and $\bar{h}$ are produced for the lemma.

For each $i\in J$, $\bigcup_{B\in I} U_B\times g_i(B)$ is a subsheaf of $S(U'_{m+1}, A_{n_k})$ isomorphic to $S'=\bigcup_{B\in I} U_B\times B$. Restricting both to the part over K_{f_i} we get a subsheaf $\bar{S}^i=\bigcup_{B\in I} U_B^i\times g_i(B)$ of $S(K_{f_i}, A_{n_k})$ and an isomorphism

$$\bar{h}_i: \bar{S}^i\cong S^i\cong\pi^{-1}(K_{f_i}).$$

For $(y, b)\in\bar{S}^i$, say $y\in U_B^i$ and $b\in g_i(B)$, $\bar{h}_i(y, b)=\tau^B_{g_i^{-1}(b)}(y)$. Since the above holds for each $i\in J$, $\bar{S}=\bigcup_{i\in J}\bar{S}^i$ is a subsheaf of $S(U'_{m+1}, A_{n_k})$ and $\bar{h}=\bigcup_i \bar{h}_i$ is an isomorphism of $\bar{S}\cong\pi^{-1}(U'_{m+1})$.

Now we check the hypothesis of Lemma 3.4. Fix $0^*\in J$. Then $g_{0*}f_{0*}=g_i f_i$ for all $i\in J$ and $g_{0*}f_{0*}$ embeds $A_m\to A_{n_k}$. Now, suppose $y\in K_{f_i}$; then $y\in K_{f_i}^B$ for some $B\in I$ where $f_i(A_m)\subseteq B$. Since, $K_{f_i}^B\subseteq U_B$, $y\in U_B^i$. Thus, $S_y^i\supseteq\{y\}\times B\supseteq\{y\}\times f_i(A_m)$. Therefore,

$$\bar{S}_y^i\supseteq\{y\}\times g_i(B)\supseteq\{y\}\times g_i f_i(A_m)=\{y\}\times g_{0*}f_{0*}(A_m).$$

Hence, $\{y\}\times g_{0*}f_{0*}(A_m)\subseteq\bar{S}_y$ for each $y\in U'_{m+1}$.

Finally, we must check that $\bar{h}(y, g_{0*}f_{0*}(b))=\sigma_b(y)$ for all $y\in U'_{m+1}$ and $b\in A_m$. Suppose $y\in K_{f_i}$ for some $i\in J$. Then $y\in K_{f_i}^B$ for some $B\in I$ with $f_i(A_m)\subseteq B$. By the definition of $K_{f_i}^B$, $\sigma_b(y)=\tau^B_{f_i(b)}(y)$. Properties of $\bar{h}_i$ and the definition of $\bar{h}$ yield

$$\begin{aligned}\tau^B_{f_i(b)}(y)&=\bar{h}_i(y, g_i f_i(b))\\&=\bar{h}(y, g_{0*}f_{0*}(b))\end{aligned}$$

as desired.

Now, by Lemma 3.4, $\pi^{-1}(N')$ admits A_{n_k}-coordinates. Hence, we have constructed a clopen neighborhood N' of the point x with type $m=n-k$ such that $\pi^{-1}(N')$ admits A_{n_k}-coordinates. That is (*) holds for k. Thus, (*) follows by induction and this completes the proof of 8.1.

Since the dual sheaf of a countable CA_1 is partitionable, the following is a special case of 8.1.

COROLLARY 8.2. *Every countable CA_1 with finite degree n is isomorphic to a $CA_1\Gamma(X, S)$ where S is a subsheaf of a constant A_{n*}-sheaf over a Boolean space X.*

Representations as algebras of continuous functions can be obtained from 8.2 and 2.5.

COROLLARY 8.3. *For every countable CA_1M with finite degree there is a Boolean space X, a finite simple CA_1A, and a family $\{X_B: B\in S(A)\}$ of closed subsets of X such that M is isomorphic to the algebra of all continuous functions from X into A for which the values on X_B belong to B for every $B\in S(A)$.*

§9. Decidability

Corresponding to the representation given in 8.2 there is a decidability result. A relationship between sectional representations and decidability was given in [2]. In particular, an immediate consequence of 2.5 in [2] is

THEOREM 9.1. *If A is a finite simple CA_1 and $\mathscr{K}(A)$ is the class of all $\Gamma(X, S)$ where S is a subsheaf of $S(X, A)$ for some Boolean space X, then $\mathscr{K}(A)$ has a decidable theory.*

From 9.1 and 8.2

Theorem 9.2. *For each $n<\omega$, $\mathrm{Th}(V_n)$ is decidable.*

Proof. V_n is characterized by a single identity ε_n in addition to the CA_1 axioms. (Cf., Monk [4]). By 8.2 the nontrivial countable members of V_n are (up to isomorphism) the countable members of $\mathscr{K}(A_{n*})$ that satisfy ε_n. By 9.1, $\mathscr{K}(A_{n*})$ is decidable. Hence, the theory of the nontrivial members of V_n, being a finite extension of $\mathrm{TH}\,\mathscr{K}(A_{n*})$, is also decidable. It follows that $\mathrm{Th}\,V_n$ is decidable.

§10. Sectional structures and reduced products

The variety V_1 is essentially the class of all Boolean algebras. It is known that every member of V_1 is elementarily equivalent to a reduced power of A_1. An example is given below of a subsheaf S of a constant A_2-sheaf with the property that $\Gamma(X, S)$ is not elementarily equivalent to a reduced power or limit power of A_2. Moreover the structure is not elementarily equivalent to any reduced product of simple CA_1's.

Let $\phi(x)$ denote the formula

$$\exists z_0, z_1 (cz_0 = x \wedge cz_1 = x \wedge z_0 \cdot z_1 = 0 \wedge z_0 + z_1 = x).$$

Consider the Horn sentence θ:

$$\exists x, y, z_0, z_1 \, \forall z [cx = x \wedge cy = y \wedge x \cdot y = 0 \wedge x + y = 1 \\ \wedge (z \cdot x = z \rightarrow cz = z) \wedge z_0 \cdot z_1 = 0 \wedge z_0 + z_1 = y \wedge cz_0 = y \wedge cz_1 = y].$$

The sentence θ says there is a closed element x such that every element contained in x is closed and ϕ is satisfied by $-x$.

LEMMA 10.1. *θ holds in every simple $CA_1 A$.*

Proof. If A is a two element CA_1, let $x=1$ and $y=z_0=z_1=0$. If A has more than two elements, choose $x=0$, $y=1$, z_0 any element in A different from 0, 1, and $z_1 = -z_0$.

Suppose X is a Boolean space dual to the BA of all finite-cofinite subsets of ω. To be precise take $X = \{0, 1, \ldots\} \cup \{\infty\}$ where ∞ is the only non-isolated point. Let $B_0 = \Gamma(X, S)$ where $S = (X \times A_1^*) \cup \{0, 1, \ldots\} \times A_2$ is the subsheaf of $S(X, A_2)$ whose stalk is A_2 at a point $\neq \infty$ and whose stalk at ∞ is the unique 2 element subalgebra A_1^* of A_2. B_0 can be described in non sheaf-theoretic terms (up to isomorphism) as an atomic CA_1 with a denumerable number of c-atoms such that (i) every c-atom contains exactly 2 atoms and (ii) every element is a sum of a finite or cofinite set of atoms.

THEOREM 10.2. *B_0 is not elementarily equivalent to a reduced product of simple CA_1's.*

Proof. By 10.1 and the fact reduced products preserve Horn sentences, it is enough to show θ fails in B_0. Suppose θ holds in B_0. Then there exist a closed $x \in B_0$ such that $\{z \in B_0 : z \leqslant x\}$ is discrete. It must be the case that $x=0$ so $1=-x$ satisfies ϕ in B_0. This is clearly impossible because S_∞ has two elements. Hence θ can not hold in B_0.

As a consequence of 10.2 the theory of all CA_1's is not the same as the theory of the class of all reduced products of simple CA_1's.

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