COLLOQUIUM MATHEMATICUM

VOL. XXXIV 1976 FASC. 2

COMPLETE AND MODEL-COMPLETE THEORIES OF MONADIC ALGEBRAS

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0. Introduction. This paper begins an investigation of the elementary theories of monadic algebras, also known as one-dimensional cylindric (or polyadic) algebras. Elementary types of Boolean algebras have been completely described by Tarski [11]. On the other hand, a complete description in the case of two-dimensional and higher dimensional cylindric or polyadic algebras is impossible since these theories are undecidable (cf. Henkin and Tarski [8] and Comer [3]). The equational theories of monadic algebras were investigated and completely described by Monk [10]. We will show there are 2^{ω} elementary types of monadic algebras and investigate certain natural complete theories.

In a recent paper [9], Macintyre gave a sufficient condition for the model-completeness of the theory of the structure of sections of a sheaf of rings. This condition is extended in Section 2 to cover structures that occur in algebraic logic. In Section 3 these results are applied to show that, for each equational class of monadic algebras, the theory of its non-trivial members has a model-companion. Axioms are given for these theories and they are shown to be decidable and ω-categorical. We assume the reader is familiar with [9].

1. Sheaf notation. Let L be a first-order language. A *sheaf of L-structures* is a triple (X, S, π), where

(i) X and S are topological spaces;

(ii) π is a local homeomorphism from S onto X;

(iii) for each $x \in X$, $\pi^{-1}(x) = S_x$ is the universe of an L-structure S_x;

(iv) for each non-logical symbol of L, the natural interpretation on S, that is induced by the interpretation on each S_x, is continuous.

See [9] for a more precise formulation of (iv). If X or π is understood from the context, we drop it from the notation. The L-structures S_x are called the *stalks* of the sheaf. (X, S) is a *sheaf of models of a theory T* if S_x is a model of T for each $x \in X$. We assume that X is a Boolean space throughout the paper.

A *section* of a sheaf (X, S, π) is a continuous map $\sigma\colon X \to S$ such that $\pi\sigma$ is the identity on X. The subset of $\prod_{x\in X} S_x$ that consists of all sections is denoted by $\Gamma(X, S)$. Condition (iv), in its precise formulation, implies that $\Gamma(X, S)$, with operations and relations inherited from the product, is an L-substructure of $\prod_{x\in X} S_x$. If A is an L-structure (topologically, a discrete space), (X, S, π) is a *constant A-sheaf* if $S = X\times A$, with the product topology, and π is the projection. If (X, S) is a constant A-sheaf, we denote the L-structure $\Gamma(X, S)$ of sections by $\Gamma(X, A)$.

For a sheaf (X, S) of L-structures we refer to $\mathrm{Th}\{S_x\colon x\in X\}$ as the *stalk theory* and to $\mathrm{Th}\big(\Gamma(X, S)\big)$ as the *section theory* of the sheaf.

An L-theory T is *positively model-complete* if every L-formula is equivalent, relative to T, to a positive existential formula (see [9]).

2. Conditions for model-completeness. Consider the following conditions:

(A) X is a Boolean space with no isolated points.

(B) T is a positively model-complete theory.

(C′) L includes two non-logical constants 0 and 1. Also, there exist two L-terms $s(v_0, v_1)$ and $p(v_0, v_1)$ and an atomic formula $\Omega(v_0)$, having one free variable v_0, in which 0 and 1 do not occur. The theory T includes the following sentences:

$$0\neq 1,\quad s(0,0)=0,\quad s(0,1)=1,\quad s(1,0)=1,\quad s(1,1)=1,$$
$$p(1,1)=1,\quad p(1,0)=0,\quad p(0,1)=0,\quad p(0,0)=0,$$
$$(\forall v_0)\big(p(v_0,1)=v_0\big),\quad (\forall v_0)\big(p(v_0,0)=0\big),\quad (\forall v_0)\big(\Omega\leftrightarrow v_0=0\vee v_0=1\big).$$

Condition (C′) is more general than conditions (C) and (D) given in [9]. The following is a modification of Macintyre's Theorem 2:

THEOREM 2.1. *For a sheaf of L-structures that satisfies* (A), *if the stalk theory satisfies* (B), (C′) *and is complete, then the section theory is model-complete.*

A proof of this theorem can be constructed from a careful analysis of the argument in [9]. Conditions (C) and (D) in [9] are used to code up clopen sets in the rings $\Gamma(X, S)$ by idempotent elements. Condition (C′) also allows us to do this.

For the atomic formula Ω in (C′), we call $\sigma\in\Gamma(X, S)$ an *Ω-element* if $\Gamma(X, S)\models\Omega[\sigma]$, i.e., $S_x\models\Omega[\sigma(x)]$ for each $x\in X$. By (C′), the Ω-elements of $\Gamma(X, S)$ are precisely the characteristic functions of clopen subsets of X. Let $\bar{s}$ and $\bar{p}$ denote the operations on $\Gamma(X, S)$ induced by the L-terms s and p from condition (C′). The set of all Ω-elements with $\bar{s}$ as sum and $\bar{p}$ as product is a Boolean algebra that is isomorphic to the

Boolean algebra of all clopen subsets of X by using the characteristic function relationship. We leave the straightforward details to the reader.

The proofs of Theorems 3, 4, and 5 in [9] can also be modified, along these lines, to give a general result.

THEOREM 2.2. *If L is an operational language (i.e., there are no non-logical relation symbols), Theorem 2.1 remains valid when we drop the assumption that the stalk theory is complete.*

THEOREM 2.3. *Suppose L contains only non-logical operation symbols and T is an L-theory that satisfies* (B) *and* (C′). *Let $\mathscr{C}$ be the class of all $\Gamma(X, S)$, where S is a sheaf of models of T, and X is a Boolean space with no isolated points. Then* $\mathrm{Th}(\mathscr{C})$ *is model-complete. The restriction to operational languages can be dropped if T is a complete theory.*

Remark. The connection between condition (C′) and Macintyre's (C) and (D) can be seen by using $v_0 \cdot v_0 = v_0$ for Ω, $v_0 \cdot v_1$ for p, and $v_0 + v_1 - v_0 \cdot v_1$ for s in condition (C′). In the next section we apply these results to theories of monadic algebras; a situation not covered by Macintyre's original theorems.

3. Model-complete theories of monadic algebras. A *monadic algebra* is a structure $\langle A, +, \cdot, -, 0, 1, c\rangle$, where $\langle A, +, \cdot, -, 0, 1\rangle$ is a Boolean algebra (BA), and c is a quantifier on this BA, i.e., $c0 = 0$, $x \leqslant cx$ and $c(x \cdot cy) = cx \cdot cy$. The element x is *closed* if $cx = x$. Denote the class of all monadic algebras by CA_1.

A *simple CA_1* is a non-trivial BA with a quantifier c such that $cx = 1$ if $x \neq 0$. For each $m = 1, 2, \ldots$, let A_m be a simple CA_1 with 2^m elements and let A_∞ be a simple, denumerable atomless CA_1. For each $m \leqslant \infty$, let V_m denote the variety generated by A_m. In [10] Monk showed that the non-trivial equational classes of CA_1's form an $(\omega+1)$-chain

$$V_1 < V_2 < \ldots < V_\infty = \mathrm{CA}_1.$$

The trivial variety of all one-element CA_1's is, of course, covered by V_1. Since the theory of the trivial variety is complete and categorical, we omit it from additional consideration.

For each $m \leqslant \infty$, let $\mathscr{C}_m$ denote the class of all CA_1's $\Gamma(X, S)$, where S is a sheaf of models of $\mathrm{Th}\{A_m\}$ and X is a Boolean space with no isolated points.

The following is a consequence of Theorem 2.3:

LEMMA 3.1. $\mathrm{Th}(\mathscr{C}_m)$ *is model-complete for each* $m \leqslant \infty$.

Proof. Condition (C′) holds using $+$ for s, $\cdot$ for p, and $cv_0 = v_0$ for Ω. Consider (B). It is clear that $\mathrm{Th}\{A_m\}$ is model-complete if $m < \infty$. $\mathrm{Th}\{A_\infty\}$ is model-complete by the same standard argument that works for atomless BA's. In any simple CA_1, $a \neq b$ is equivalent to $c(a \oplus b) = 1$, where $\oplus$

denotes the symmetric difference. Hence, the negation of a CA_1 equation is equivalent to an equation relative to the theory of simple CA_1's. It follows that $\mathrm{Th}\{A_m\}$ is positively model-complete.

Let X_0 denote the Cantor discontinuum 2^ω. The following result is immediate from Theorems 1.2 and 1.3 of [5] and from the fact that every two atomless BA's are elementarily equivalent.

LEMMA 3.2. $\mathrm{Th}(\mathscr{C}_m) = \mathrm{Th}\{\Gamma(X_0, A_m)\}$ *for each* $m \leqslant \infty$.

For L-theories T and T^* we say that T^* is a *model-companion* of T if $T \subseteq T^*$, T^* is model-complete, and every model of T is embeddable in a model of T^*. This notion was introduced by E. Bers as a refinement of A. Robinson's notion of model completion. For the basic facts, see [1] and [6]. If a model completion of a theory exists, then it is a model-companion. If a theory has a model-companion, it is unique. Finally, let T have a model-companion T^*; then T^* is a model completion if and only if the class of all models of T has the amalgamation property.

Let T_m denote the theory of the non-trivial members of V_m for $m \leqslant \infty$. The main result of this section is

THEOREM 3.1. $\mathrm{Th}(\mathscr{C}_m)$ *is a model-companion of* T_m *for each* $m \leqslant \infty$. *It is a model completion for* $m = 1, 2$ *and* ∞ *but not otherwise.*

Proof. By Lemma 3.1, $\mathrm{Th}(\mathscr{C}_m)$ is model-complete so, to verify the first assertion, it remains to show that every non-trivial member A of V_m is embeddable in a model of $\mathrm{Th}(\mathscr{C}_m)$. In view of the Henkin embedding theorem [7] or the fact that an algebra is embeddable in an ultraproduct of its finitely generated subalgebras, it is enough to consider A finitely generated. But every finitely generated CA_1 is finite. The sectional representation results [2], restricted to CA_1's, imply that every non-trivial finite A in V_m is isomorphic to a finite product $\prod_{i<n} B_i$, where each B_i is embeddable in A_m. If we use the natural embedding $A_m \to \Gamma(X_0, A_m)$, each B_i is embeddable into $\Gamma(X_0, A_m)$. Hence $\prod_{i<n} B_i$ is embeddable in $\Gamma(X_0, A_m)^n$. By the dual sheaf theory [2] for CA_1's, finite products correspond to sums of sheaves, so $\Gamma(X_0, A_m)^n \cong \Gamma(Y, S)$, where Y is homeomorphic to a disjoint union of n Cantor spaces (and hence has no isolated points) and each stalk of S is isomorphic to A_m. Hence, $\prod_{i<n} B_i$ is embeddable in $\Gamma(X_0, A_m) \in C_m$ as desired. The second assertion in the theorem follows from the first and the fact that V_1, V_2 and V_∞ are the only varieties of CA_1's with the amalgamation property.

Remark. Many model-complete theories of n-dimensional cylindric algebras (CA_n's) and polyadic algebras (PA_n's) $(1 < n < \omega)$ can be obtained from the results in Section 2. But since the amalgamation property fails for CA_n's and PA_n's with $1 < n < \omega$ (see [4]), the theories of non-trivial CA_n's and PA_n's do not have a model completion for $1 < n < \omega$.

4. Properties of $\mathrm{Th}(\mathscr{C}_m)$. In this section we find axioms for $\mathrm{Th}(\mathscr{C}_m)$ and prove that the theory is ω-categorical and decidable.

The following properties can be expressed as first-order statements:

(1) the axioms for non-trivial CA_1's;

(2) the BA of all closed elements is atomless;

(3) $(\forall v_0)\big[v_0 \neq 0 \rightarrow (\exists v_1)\big(v_1 \leqslant v_0 \wedge cv_1 = cv_0 \wedge c\big(v_0\cdot(-v_1)\big) = cv_0\big)\big]$.

For $m < \infty$,

$$(4)_m \qquad 0 = \prod_{i<j<n} c(v_i \oplus v_j), \qquad \text{where } n = 2^m-1;$$

$$(5)_m \qquad (\forall v_0)\big[cv_0 = v_0 \wedge v_0 \neq 0 \rightarrow (\exists v_1)\ldots(\exists v_{n-1})\big(\bigwedge_{i<n} cv_i = v_0 \wedge \bigwedge_{i<j<n} c(v_i\oplus v_j) = v_0\big)\big], \qquad \text{where } n = 2^m-1.$$

For $m < \infty$, V_m is characterized by the CA_1 axioms plus $(4)_m$ (see Monk [10]). Notice that a simple CA_1 satisfies $(4)_m$ if and only if it has at most 2^m elements, and that it satisfies $(5)_m$ if and only if it contains at least 2^m elements. Thus, $(4)_m$ and $(5)_m$ characterize A_m among the simple CA_1's. Similarly, (3) holds in a simple CA_1 if and only if it is atomless.

The following axioms were obtained by lifting the above-given properties of the stalks:

THEOREM 4.1. (i) *Statements* (1), (2) *and* (3) *provide axioms for* $\mathrm{Th}(\mathscr{C}_\infty)$.

(ii) *For* $m < \infty$, (1), (2), $(4)_m$ *and* $(5)_m$ *is a set of axioms for* $\mathrm{Th}(\mathscr{C}_m)$.

Proof. (i) Clearly, (1) and (2) hold in $\mathscr{C}_\infty$. Suppose $\Gamma(X, S)\in\mathscr{C}_\infty$, $\sigma\in\Gamma(X, S)$ and $\sigma\neq 0$. Then $\|\sigma\| = \{x\in X\colon \sigma(x)\neq 0_x\}$ is a non-empty clopen set. For each $x\in\|\sigma\|$, S_x is atomless, so there exist $\tau_x\in\Gamma(X, S)$ with $\tau_x(x) < \sigma(x)$. By a standard "globalization" argument, there is a $\tau\in\Gamma(X, S)$ such that $\|\sigma\| = \|\tau\|$ and $\tau(y) < \sigma(y)$ for all $y\in\|\sigma\|$. Thus, (3) holds.

Conversely, suppose A is a model of (1), (2) and (3). By [2], $A\cong\Gamma(X, S)$, where S is a sheaf of simple CA_1's. Since (2) holds, X has no isolated points. $\Gamma(X, S)$ will belong to $\mathscr{C}_\infty$ if each stalk S_x is atomless. Suppose $0\neq s\in S_x$. Choose $\sigma\in\Gamma(X, S)$ so that $\sigma(x) = s$. We have $\sigma\neq 0$, since $s\neq 0$. By (3), there is a $\tau\in\Gamma(X, S)$ such that $\tau\leqslant\sigma$, $c\tau = c\sigma$, and $c\big(\sigma\cdot(-\tau)\big) = c\sigma$. Evaluating these equations at x yields $0_x < \tau(x) < s$. Hence S_x is atomless. Thus, every model of (1), (2) and (3) belongs to $\mathscr{C}_\infty$ (up to isomorphism).

The proof of (ii) is similar.

As a corollary to Lemma 3.2 and Theorem 4.1 we have

COROLLARY 4.1. $\mathrm{Th}(\mathscr{C}_m)$ *is decidable for* $m\leqslant\infty$.

The connection between reduced products, limit powers, and the structures $\Gamma(X, A)$ was pointed out by Macintyre in [9]. This connection allows us to use results of Waszkiewicz and Węglorz from [12] and [13]. In general, for any A, $\Gamma(X, A)$ is the limit power $A^X | F_X$, where F_X denotes the filter on $X \times X$ generated by all equivalence relations that correspond to clopen partitions of X. For the Cantor space X_0, $2^{X_0} | F_{X_0}$ is an infinite atomless BA, so Theorem 1 in [12] implies that $\Gamma(X_0, A_m) \equiv (A_m)_D^\omega$, where D is the filter of all cofinite subsets of ω. Since 2_D^ω is atomless, Lemma 3.2, together with Corollary 2.2 in [13], gives

THEOREM 4.2. $\mathrm{Th}(\mathscr{C}_m) = \mathrm{Th}\{\Gamma(X_0, A_m)\} = \mathrm{Th}\{A_m\}_D^\omega$ *is* ω-*categorical for* $m \leqslant \infty$.

5. The number of elementary types of monadic algebras. There are 2^ω complete theories of CA_1's constructed in this section.

Denote the BA of all closed elements of a CA_1 A by $Z(A)$. An atom of $Z(A)$ is a *c-atom*. For a c-atom y of A, let At_y denote the set of all atoms $a \in A$, $a \leqslant y$.

For an atomic CA_1 A, introduce a function $f^A \in {}^{\omega \sim 1}2$ defined for $n \in \omega \sim 1$ by

$$f^A(n) = \begin{cases} 1 & \text{if } \Sigma^A\{y \in Z(A)\colon y \text{ is a } c\text{-atom}, |\mathrm{At}_y| = n\} \text{ exist}, \\ 0 & \text{otherwise}. \end{cases}$$

The following lemma produces the desired examples:

LEMMA 5.1. *For each subset* Σ *of* $\omega \sim 1$, *there exists an atomic* CA_1 A_Σ *such that* $f^{A_\Sigma}(n) = 1$ *if and only if* $n \in \Sigma$.

Proof. Partition a countable infinite set X into an infinite number of pairwise disjoint sets $X_1, X_2, \ldots$ Partition each X_n into an infinite number of disjoint sets Y_{na}, $a = 1, 2, \ldots$, where $|Y_{na}| = n$. Let A denote the complete atomic BA of all subsets of X, and B the BA that consists of all finite and cofinite subsets of X. For a subset Σ of $\{1, 2, \ldots\}$, let A_Σ denote the Boolean subalgebra of A generated by $B \cup \{X_n\colon n \in \Sigma\}$.

The following property of A_Σ is useful.

(*) *For each* k, $X_k \in A_\Sigma$ *if and only if* $k \in \Sigma$.

For $a, b \in A_\Sigma$, write $a \sim_\omega b$ if $a \oplus b$ is finite. Since there are an infinite number of X_i's, $X_k \in A_\Sigma$ implies $k \in \Sigma$ in view of the fact that, for every $a \in A$, $a \sim_\omega b$ for some b in the subalgebra of A generated by $\{X_n\colon n \in \Sigma\}$. Thus (*) holds.

We introduce a closure operation c on A_Σ by defining, for $Y \in A_\Sigma$,

$$c(Y) = \bigcup \{Y_{na}\colon Y_{na} \cap Y \neq 0\}.$$

For each atom $a \in A_\Sigma$, $c(a) = Y_{na}$, where $a \leqslant Y_{na}$, and the center $Z(A_\Sigma)$ of A_Σ is atomic with $\{Y_{na}\colon n, a \in \omega \sim 1\}$ as the set of atoms. Each X_n, with $n \in \Sigma$, is closed, so $Z(A_\Sigma)$ is the Boolean subalgebra of A_Σ generated by

$$\{Y_{na}\colon n, a \in \omega \sim 1\} \cup \{X_n\colon n \in \Sigma\}.$$

For each n,

$$\{y \in Z(A_\Sigma): y \text{ is a } c\text{-atom}, |\mathrm{At}_y| = n\} = \{Y_{n\alpha}: \alpha \in \omega \sim 1\}$$

and, for $n \in \Sigma$,

$$X_n = \Sigma_\alpha^{A_\Sigma} Y_{n\alpha}.$$

Thus, $f^{A_\Sigma}(n) = 1$ whenever $n \in \Sigma$.

On the other hand, suppose $\Sigma^{A_\Sigma} Y_{n\alpha} = a$ exists in A_Σ for some n. Since $cY_{n\alpha} = Y_{n\alpha}$ for each α, we have

$$ca = \Sigma^{A_\Sigma} cY_{n\alpha} = \Sigma^{A_\Sigma} Y_{n\alpha} = a, \quad \text{i.e., } a \in Z(A_\Sigma).$$

$X_n \subseteq a$, since $Y_{n\alpha} \subseteq a$ for all α. It follows that $a = X_n$, since no atom $Y_{m\beta}$ of $Z(A_\Sigma)$ is contained in a whenever $m \neq n$. Hence, if $f^{A_\Sigma}(n) = 1$, $\Sigma_\alpha^{A_\Sigma} Y_{n\alpha}$ exists in A_Σ and equals X_n. By (*) we have $n \in \Sigma$ as desired.

For each $n \in \omega \sim 1$, let φ_n denote the sentence that says: the sum of the set of c-atoms that contains exactly n atoms exists.

For each $\Sigma \subseteq \omega \sim 1$, let T_Σ denote the set of all sentences derivable from the axioms for atomic CA_1's and $\{\varphi_n: n \in \Sigma\} \cup \{\neg\varphi_n: n \notin \Sigma\}$. By Lemma 5.1, A_Σ is a model of T_Σ. Since it is clear that no model of T_Σ is a model of $T_{\Sigma'}$ whenever $\Sigma \neq \Sigma'$, we obtain the desired result.

THEOREM 5.2. *The theory of atomic CA_1's contains 2^ω complete extensions.*

REFERENCES

[1] J. Barwise and A. Robinson, *Completing theories by forcing*, Annals of Mathematical Logic 2 (1970), p. 119-142.

[2] S. Comer, *A sheaf-theoretic duality theory for cylindric algebras*, Transactions of the American Mathematical Society 169 (1972), p. 75-87.

[3] — *Finite inseparability of some theories of cylindrification algebras*, Journal of Symbolic Logic 34 (1969), p. 171-176.

[4] — *Classes without the amalgamation property*, Pacific Journal of Mathematics 28 (1969), p. 309-318.

[5] — *Elementary properties of structures of sections*, Boletín de la Sociedad Matemática Mexicana (to appear).

[6] P. Eklof and G. Sabbagh, *Model-completions and modules*, Annals of Mathematical Logic 2 (1971), p. 251-295.

[7] L. Henkin, *Some interconnections between modern algebra and mathematical logic*, Transactions of the American Mathematical Society 74 (1953), p. 410-427.

[8] — and A. Tarski, *Cylindric algebras. Lattice theory*, Proceedings of the Symposium in Pure Mathematics 2 (1961), p. 83-113.

[9] A. Macintyre, *Model-completeness for sheaves of structures*, Fundamenta Mathematicae 81 (1973), p. 73-89.

[10] D. Monk, *On equational classes of algebraic versions of logic. I*, Mathematica Scandinavica 27 (1970), p. 53-71.

[11] A. Tarski, *Arithmetical classes and types of Boolean algebras*, Bulletin of the American Mathematical Society 55 (1949), p. 64.
[12] J. Waszkiewicz and B. Węglorz, *Some models of theories of reduced powers*, Bulletin de l'Académie Polonaise de Sciences, Série des sciences mathématiques, astronomiques et physiques, 16 (1968), p. 683-685.
[13] — *On ω_0-categoricity of powers*, ibidem 17 (1969), p. 195-199.

Reçu par la Rédaction le 24. 9. 1974