HOUSTON JOURNAL OF MATHEMATICS, Volume 2, No. 3, 1976.

MULTI-VALUED LOOPS, GEOMETRIES, AND ALGEBRAIC LOGIC

Stephen D. Comer *

ABSTRACT. A notion of multi-valued loop is introduced. The multi-valued loop associated with a finite projective geometry with at least 3 points on a line is isomorphic to a "quotient" of an ordinary loop modulo a special equivalence relation. An application to algebraic logic is given.

A multi-valued loop is a system $(A, \cdot, e)$ where $e \in A$ and for each a, $b \in A$, $a \cdot b$ is a nonempty set of elements of A such that

(i) for each a,b there exist x,y such that $b \in a \cdot x$ and $b \in y \cdot a$,

(ii) for every a, $a \cdot e = \{a\} = e \cdot a$, and

(iii) for every a there exist unique x,y such that $e \in a \cdot x$ and $e \in y \cdot a$.

Multi-valued loops can be viewed as special multi-groupoids as introduced in Bruck [1]. Other multi-valued concepts that have appeared in the literature, e.g., the notion of multigroup due to Dresher and Ore [2] and the notion of cogroup due to Eaton [3], seem to be unrelated to the notion considered here. However, examples of multi-valued loops readily arise in the study of groups, geometry, and algebraic logic to mention a few areas. This paper will deal with multi-valued loops associated with projective geometries and with a relationship between multi-valued loops and representations of atomic structures of certain 3-dimensional cylindric algebras. Our main result states that the multi-valued loop associated with a finite projective geometry that has at least three points on a line can be obtained as a quotient of an ordinary loop modulo a special equivalence relation. In the last section we apply this result to partially answer a question concerning algebraic logic raised by Monk in [5]. The author wishes to acknowledge the valuable suggestions of the referee concerning the presentation of the main construction.

1. A projective geometry is a system $G = (P, L)$ where P is a non-empty set (called the "points" of G), L is a non-empty collection of subsets of P (called the "lines" of G) and the following axioms hold:

*This paper was supported in part by a grant from the Citadel Development Foundation.

(G_1) each line contains at least three points;

(G_2) each pair of distinct points p and q lie on a unique line $\overline{pq}$;

(G_3) if p, q, r, and s are distinct points and $\overline{pq}$ and $\overline{rs}$ have a common point, then $\overline{pr}$ and $\overline{qs}$ have a common point.

Notice that the notion of projective geometry allows dimension 1,2, or higher; e.g., see Seidenberg [6].

A multi-valued loop can be constructed from a projective geometry G in the following way. Choose $e \notin P$ and let $A = \{e\} \cup P$. Denote by L_G the system (A,*,e) where $p * e = e * p = \{p\}$ for all $p \in A$ and for $p,q \in P$

$$p*q = \begin{cases} \{e,p\} & \text{if } p = q \\ pq - \{p,q\} & \text{if } p \neq q \end{cases}$$

Clearly L_G is a multi-valued loop.

A multi-valued loop can also be constructed from an ordinary loop via quotient construction. Suppose $A = (A, \cdot, e)$ is a loop and θ is an equivalence relation on A. The product XY of two blocks (i.e., equivalence classes) of θ is $XY = \{x \cdot y \in A: x \in X, y \in Y\}$.

A block $B \subseteq XY$ is called (X,Y)-*projective* if for every $x \in X$, $xy \in B$ for some $y \in Y$ and, similarly, for every $y \in Y$ there is an $x \in X$ with $xy \in B$. We call θ *special* if $\{e\}$ is one of the θ-blocks, and every product XY of θ-blocks is a union of (X,Y)-projective blocks. Special equivalence relations need not be congruence relations.

PROPOSITION 1. *For a loop $A = (A,\cdot,e)$ and a special equivalence relation θ on A, let $A/\theta = (A/\theta, *, E)$ where A/θ is the set of all θ-blocks of A, $E = e/\theta = \{e\}$, and for blocks X,Y,B*

$$B \in X * Y \text{ iff } B \subseteq XY.$$

Then A/θ is a multi-valued loop. We refer to A/θ as a quotient of A.

The proof is routine. To illustrate we verify (iii). For blocks X,Y, note that $\{e\} \subseteq XY$ iff $e = xy$ for some $x \in X$ and $y \in Y$. For $x \in X$, by (iii) for A, $e = xy$ for some $y \in A$; thus, $\{e\} \subseteq XY$ for $Y = y/\theta$. If also $\{e\} \subseteq XY'$, then since $\{e\}$ is (X,Y')-projective, $e = xy'$ for some $y' \in Y'$. By (iii) for A, this implies that $y = y'$. Hence $Y = Y'$.

Not every multi-valued loop is isomorphic to a quotient A/θ of an ordinary loop.

PROPOSITION 2. *There does not exist a loop A and a special equivalence relation θ on A such that A/θ is isomorphic to the 3 element multi-valued loop whose multiplication table is given below.*

·	1	2	3
1	1	2	3
2	2	{1,3}	3
3	3	2	{1,2}

PROOF. Every quotient A/θ satisfies the following property (see Proposition 1 for notation):

$$(1)\ \forall B[\forall X(B \in B * X \rightarrow X = E) \rightarrow \forall X(B \in X * B \rightarrow X = E)].$$

Fix $B \in A/\theta$ and suppose that, for all X,B $\subseteq$ BX implies X = E; in addition, assume that B $\subseteq$ YB. We must show that Y = E. Choose b $\in$ B. Since B is (Y,B)-projective, yb $\in$ B for some y $\in$ Y. Now yb = bx for some x. If $X = x/\theta$, then yb $\in$ B $\cap$ BX; thus, since θ is special, B $\subseteq$ BX. By hypothesis, this implies X = E = {e}; hence, x = e and so yb = bx = be = b. Since A is a loop, this implies y = e. Hence Y = E and so (1) holds in A/θ.

The 3 element multi-valued loop above has the property asserted in the proposition since it fails to satisfy (1). This can be seen by choosing B = 3; clearly $\forall x(3 \in 3 \cdot x \rightarrow x = 1)$ but $3 \in 2 \cdot 3$ while $2 \neq 1$.

2. Our main result is the construction of a loop $\boldsymbol{L} = (L,\cdot,q)$ and a special equivalence relation θ on L from a finite geometry $\boldsymbol{G}$ so that $\boldsymbol{L}_{\boldsymbol{G}} \cong \boldsymbol{L}/\theta$.

We first consider the case where $\boldsymbol{G}$ is a finite geometry with $N \geqslant 4$ points on each line. Order the set P of points of $\boldsymbol{G}$ as $p_1, p_2, \ldots$. For each p_i choose a set $A_i = \{a_{i,1}, \ldots, a_{i,N-2}\}$ with N-2 elements in such a way that $A_i \cap A_j = 0$ whenever $i \neq j$. Choose q distinct from all $a_{i,k}$'s and set $L = \{q\} \cup \cup_i A_i$. Let θ be the equivalence relation on L that has {q} and the A_j's as its equivalence classes.

For $i \neq j$, suppose $\overline{p_i p_j} = \{p_{h_1}, \ldots, p_{h_N}\}$ where $h_1 < h_2 < \ldots < h_N$. Let $\overline{p_i p_j}^*$ result from permuting $(p_{h_1}, \ldots, p_{h_N})$ cyclically to begin with p_i, and then deleting p_i and p_j. Let $\beta^{ij}(k)$ be the subscript of the kth term in $\overline{p_i p_j}^*$; that is,

$$\overline{p_i p_j}^* = (p_{\beta^{ij}(1)}, \ldots, p_{\beta^{ij}(N-2)}).$$

For a finite subset F of P and $p_i \in F$ we say that p_i has rank n in F if p_i is the nth element in F (under the linear ordering induced by the subscripts).

Still assuming $i \neq j$, for each $p_m \in \overline{p_i p_j} - \{p_i, p_j\}$, let $\gamma^{ij}(m)$ be the rank of p_j in $\overline{p_i p_j} - \{p_i, p_m\}$ and let $\rho^i(j)$ be the rank of p_j in $\overline{p_i p_j}$. If p_i, p_j and p_m are three distinct collinear points, notice that $\rho^i(j) = \rho^m(j)$. In addition we find that

$$(+) \qquad \gamma^{ij}(m) = \begin{cases} \rho^i(j) & \text{if } j < i,m \\ \rho^i(j) - 1 & \text{if } i < j < m \text{ or } m < j < 1 \\ \rho^i(j) - 2 & \text{if } i,m < j. \end{cases}$$

In the following, the second subscript k on $a_{i,k}$ should be regarded as an integer modulo N-1. Thus, sums and differences involving these subscripts should be calculated mod N-1.

Now we introduce a binary operation $\cdot$ on L. First let $q \cdot a = a = a \cdot q$ for every $a \in L$. Next we define

$$a_{ik} \cdot a_{i\ell} = \begin{cases} a_{i,\,k+\ell} & \text{if } k + \ell \neq 0 \pmod{N-1} \\ q & \text{if } k + \ell = 0 \pmod{N-1} \end{cases}$$

Finally, if $i \neq j$, we define $a_{ik} \cdot a_{j\ell} = a_{mn}$ where

$$(\Delta) \quad m = \begin{cases} \beta^{ij}(k + \ell - 1) & \text{if } i < j \\ \beta^{ij}(k + \ell) & \text{if } i > j \end{cases}$$

and

$$n = \gamma^{ij}(m) + k - 1 \pmod{N-1}.$$

For the case where the geometry G has exactly three points on each line the $(L, \cdot, q)$ constructed above will also be a loop; however, the equivalence relation θ is the identity in this case and so this construction does not produce the desired isomorphism $L_G \cong L/\theta$ when N = 3. A construction that works for N = 3 is similar to the above but simpler. As before order the set P of all points of G as $p_1, p_2, \ldots$; now choose pairwise disjoint two element sets $A_i = \{a_{io}, a_{i1}\}$ for each p_i. Choose q distinct from all a_{ik}'s and define L and θ as before using the present q and A_i's. The binary operation on L is now defined so that q is the identity,

$$a_{ik} \cdot a_{i\ell} = \begin{cases} q & \text{if } k \neq \ell \\ a_{ik+1} & \text{if } k = \ell \end{cases}$$

(second subscript calculated mod 2) and, if $i \neq j$, $a_{ik} \cdot a_{j\ell} = a_{mn}$ where p_m is the unique third point on $\overline{p_i p_j}$ and $n = k + \ell (\mathrm{mod}\ 2)$.

THEOREM 3. (i) *$L = (L, \cdot, q)$ is a loop;*

(ii) *θ is a special equivalence relation on L;*

(iii) $L_G \cong L/\theta$.

PROOF. We give the details of the arguments only in the case where L is constructed from a G with $N \geqslant 4$ points on a line; the arguments when $N = 3$ are similar but simpler.

(i) We must show that, given z and one of x and y, the condition that xy = z determines the other uniquely. The case that any of x,y, or z is q is immediate. Let $x = a_{ik}$, $y = a_{j\ell}$, $z = a_{mn}$. If two of i,j,m are the same all three are the same, and the conclusion is again immediate. We suppose henceforth that i,j, and m are all different.

CASE 1: a_{ik} and a_{mn} given. Since, for fixed i and m, $\gamma^{ij}(m)$ increases monotonically with j, there exists a unique $p_j \in \overline{p_i p_m} - \{p_j, p_m\}$ such that $\gamma^{ij}(m) + k - 1 = n\ (\mathrm{mod}\ N - 1)$. Similarly, with m,i, and j now given, there is a unique h such that $\beta^{ij}(h) = m$. If $i < j$ we determine ℓ uniquely from the condition that $h = k + \ell - 1\ (\mathrm{mod}\ N - 1)$, and, if $i > j$, from the condition $h = k + \ell (\mathrm{mod}\ N - 1)$.

CASE 2: $a_{j\ell}$ and a_{mn} given. We first assume that $j < m$. Then (+) becomes

$$\gamma^{ij}(m) = \begin{cases} \rho^i(j) - 1 & \text{if } i < j \\ \rho^i(j) & \text{if } i > j. \end{cases}$$

We seek i and k such that

(I) $\gamma^{ij}(m) = n - k + 1$,

(II) either $\beta^{ij}(k + \ell - 1) = m$ and $i < j$, or $\beta^{ij}(k + \ell) = m$ and $i > j$.

In view of (+) and $\rho^i(j) = \rho^m(j)$ we may rewrite (I) as

(I′) either $k = n - \rho^m(j) + 2$ and $i < j$, or $k = n - \rho^m(j) + 1$ and $i > j$.

Using (I′) to eliminate k, we are left with the condition

(II′) $\beta^{ij}(n - \rho^m(j) + \ell + 1) = m$,

which is independent of whether $i < j$ or $i > j$. Here (II′), as an equation in the known quantities j, ℓ, m and n together with the unknown i, has a unique solution for i. With this value for i, condition (I′) yields a unique solution for k.

The case $j > m$ is entirely analogous.

(ii). It suffices to show that, for $i \neq j$, A_iA_j is a union of (A_i,A_j)-projective blocks A_m. Multiplication defines a map μ: $A_i x A_j \to \cup_{m \neq i,j} A_m$. The formula ($\Delta$) shows that each m occurs for some choice of k and ℓ. The formula $n = \gamma^{ij}(m) + k - 1$ (mod $N - 1$) shows that, while keeping $k + \ell$ and therefore m fixed, by varying k we can obtain every value of n. Thus the map μ is onto. It remains to show that each A_m, for $m \neq i,j$, is (A_i,A_j)-projective. We must show that for all a_{ik} there exist $a_{j\ell}$ such that $a_{ik}a_{j\ell} \in A_m$ (and symmetrically); but this results again from the formula (Δ).

(iii) We define a function f: $L_G \to L/\theta$ as follows. For $a \in L_G$,

$$f(a) = \begin{cases} \{q\} & \text{if } a = e \\ A_i & \text{if } a = p_i \in P \end{cases}$$

It is straightforward to show that f is an isomorphism of $\boldsymbol{L_G}$ onto $\boldsymbol{L}/\theta$.

3. We now consider one of many applications of multi-valued loops to the study of cylindric algebras. An extensive study of the algebraic theory of cylindric algebras is given in Henkin, Monk, Tarski [4]; however, it would suffice for the reader to be familiar with the survey article [5]. To produce a non-representable CA_3 in [5], Monk associated an algebra with a quasigroup having 3 distinquished elements. The same construction (repeated below) gives a CA_3 for each multi-valued loop. For a multi-valued loop $\mathcal{A} = (A,\cdot,e)$ consider the structure $\mathcal{B}^{\mathcal{A}} = (S(R),\cup,\cap,\sim,0,R,c_i,d_{ij})_{i,j<3}$ where $R = \{(x_0,x_1,x_2) \in A^3\colon x_2 \in x_0 \cdot x_1\}$, (i.e., R is the graph of $\cdot$ viewed as a subset of A^3), and for $X \subseteq R$ and $i,j < 3$,

$$c_iX = \{y \in R\colon y_i = x_i \text{ for some } x \in X\},$$

and

$$d_{ij} = \begin{cases} R & \text{if } i = j \\ c_k\{(e,e,e)\} & \text{if } \{i,j,k\} = \{0,1,2\}. \end{cases}$$

The proof of the next result is routine, e.g., see Monk [5].

PROPOSITION 4. *$\mathcal{B}^{\mathcal{A}}$ is a CA_3 for every multi-valued loop $\mathcal{A}$.*

We say that a $CA_3\mathcal{B}$ is *loop representable* if there is an ordinary loop $\mathcal{A}$ such that $\mathcal{B}$ is embeddable in $\mathcal{B}^{\mathcal{A}}$. In [5], such a $\mathcal{B}$ is called an $\mathcal{A}$, q - CA_3. There is no loss by considering loops in place of quasigroups because isotopic systems induce isomorphic CA_3's and, for every specified triple in the graph of a quasigroup, there is an isotopy of the quasigroup to a loop that maps the triple to the identity triple (see Bruck [1]).

In [5] Monk also associated a $CA_3 \mathbf{A}_G$ with every projective geometry G having at least 4 points on a line. It is routine to verify that $\mathbf{A}_G \cong \mathbf{B}^L G$ by examining the atomic structures of the two algebras. The question of finding a connection between the loop representable CA_3's and the $\mathbf{A}_G$'s was raised in [5]. As an application of Theorem 3 the following shows that the finite $\mathbf{A}_G$'s are loop representable.

THEOREM 5. *For every loop A and special equivalence relation θ on A, $\mathbf{B}^{A/\theta}$ is isomorphic to a subalgebra of $\mathbf{B}^A$.*

PROOF. Suppose $A = (A,\cdot,e)$ is a loop, θ a special equivalence relation on A, and $R = \{(B_0,B_1,B_2) \in (A/\theta)^3 \colon B_2 \in B_0 * B_1\}$ is the graph of the operation * of A/θ. For $(B_0,B_1,B_2) \in R$, define

$$a_{B0,B1,B2} = \{ (x,y,z) \in A^3 \colon x \in B_0,\ y \in B_1,\ z = x \cdot y \in B_2 \}$$

Clearly, $a_{B0,B1,B2} \in B^A$ and $\{a_{B0,B1,B2} \colon (B_0,B_1,B_2) \in R\}$ is a partition of the graph of $\cdot$ into nonempty sets. Define $f \colon S(R) \to B^A$ for all $X \subseteq R$ by

$$f(X) = \cup \{a_{B0,B1,B2} \colon (B_0,B_1,B_2) \in X\}.$$

It is routine to verify that f is the desired embedding of $\mathbf{B}^{A/\theta}$ into $\mathbf{B}^A$ using the following facts.

(1) $a_{\{e\},\{e\},\{e\}} = \{(e,e,e)\}$. (Recall, $e/\theta = \{e\}$.)

For $\{i,j,k\} = \{0,1,2\}$,

(2) $d_{ij} = \cup \{a_{B0,B1,B2} \colon B_k = \{e\}\}$.

(3) $c_i a_{B0,B1,B2} = \cup \{a_{C0,C1,C2} \colon C_i = B_i\}$.

From Theorems 3 and 5 we have

COROLLARY 6. *If G is a finite projective geometry with at least four points on a line, then $\mathbf{A}_G$ (or equivalently $\mathbf{B}^L G$) is loop representable.*

REFERENCES

1. Bruck, R. H., *A Survey of Binary Systems*, Springer-Verlag, 1966.
2. Dresher, M. And O. Ore, *Theory of Multigroups*, Amer. J. Math., 60(1938), 705-733.
3. Eaton, J. E., *Theory of cogroups*, Duke Math. J., 6(1940), 101-107.
4. Henkin, L., J. D. Monk, and A. Tarski, *Cylindric Algebras, Part I*, North Holland, 1971.

5. Monk, J. D., *Connections between Combinatorial Theory and Algebraic Logic,* Studies in Algebraic Logic, Math Assoc. Amer., 1974, 58-91.
6. Seidenberg, A., *Lectures in Projective Geometry,* Van Nostrand, 1962.

The Citadel
Charleston, South Carolina 29409

Received November 14, 1975