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The Countable Chain Condition and Free Algebras

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Let A be an algebra that contains a constant 0 and a binary operation $\cdot$ among its fundamental operations. Elements $x, y \in A$ are called *disjoint* if $x \cdot y = 0$. A satisfies the *countable chain condition* (c.c.c.) if it contains no uncountable set of pairwise disjoint nonzero elements. Consider the following facts:

1. (Classical) A free Boolean algebra satisfies c.c.c.
2. (Abian [1]) A free algebra in the ring variety generated by a finite field satisfies c.c.c.

These examples suggest the question "Does c.c.c. hold in every free algebra of a quasi-primal variety?" Before this question can be answered we must know what the c.c.c. means in a universal algebra. In Section 2 we propose a general algebraic condition, called the countable separability condition (c.s.c.), and show that, in many situations, it is equivalent to c.c.c. Using c.s.c. we give an affirmative answer to the posed question (Theorem 2.9).

In Section 1 the free algebras in quasi-primal varieties are determined. The main tool for their description is the second duality result for quasi-primal varieties due to Keimel and Werner [2]. For unexplained notation and terminology see [2].

Section 1. Free Algebras in Quasi-Primal Varieties

Suppose $\mathscr{K}$ is a finite set of weakly independent quasi-primal algebras. The variety $\mathscr{V}$ generated by $\mathscr{K}$ is called a quasi-primal variety. Without loss of generality we may assume that whenever two algebras in $\mathscr{K}$ intersect the intersection is a subalgebra of both. The following construction was given as the second representation result by Keimel and Werner in [2].

Consider $\bar{\mathscr{K}} = \bigcup \mathscr{K}$ as a (finite) discrete topological space. For $R \in \mathscr{V}$ consider the set $\mathrm{Hom}(R, \mathscr{K}) = \{\varphi\colon \varphi$ is a homomorphism from R into some $A \in \mathscr{K}\}$ with the topology σ inherited from the product space $\bar{\mathscr{K}}^R$. With this topology $\mathrm{Hom}(R, \mathscr{K})$, denoted $\mathrm{Hom}_\sigma(R, \mathscr{K})$, is a Boolean space with a standard subbasis given by the collection of all sets that have the form $K(r; a) = \{\varphi \in \mathrm{Hom}(R, \mathscr{K})\colon \varphi(r) = a\}$ where $r \in R$ and $a \in \bar{\mathscr{K}}$.

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Let $\tilde{A}=\mathrm{Hom}_\sigma(R, A)$ for each $A\in S(\mathscr{K})$. $\tilde{A}$ is a closed subspace of $\mathrm{Hom}_\sigma(R,\mathscr{K})$. The collection of all isomorphisms $\alpha\colon A\to B$ where $A, B\in S(\mathscr{K})$ form an inverse semigroup H. Each $\alpha\colon A\to B$ in H induces a homeomorphism $\tilde{\alpha}\colon \tilde{A}\to\tilde{B}$ by $\tilde{\alpha}(\varphi)=\alpha\circ\varphi$ where $\varphi\in\tilde{A}$. This action of H makes $\mathrm{Hom}_\sigma(R,\mathscr{K})$ into a Boolean H-space.

Now suppose X is a Boolean space, for each $A\in S(\mathscr{K})$ there is a closed subspace $\tilde{A}$ of X, and for each $\alpha\colon A\to B$ in H there is a homeomorphism $\tilde{\alpha}\colon\tilde{A}\to\tilde{B}$ such that $\tilde{\alpha}\tilde{\beta}=\widetilde{\alpha\beta}$ for all $\alpha,\beta\in H$ (i.e., X is a Boolean H-space). Let $\mathrm{Adm}(X,\mathscr{K})$ denote the set of all continuous functions $f\colon X\to\mathscr{K}$ such that

(i) $f(\tilde{A})\subseteq A$ for all $A\in S(\mathscr{K})$,

(ii) $f(\tilde{\alpha}x)=\alpha f(x)$ for every $x\in\tilde{A}$ and isomorphism $\alpha\colon A\to B$ in H.

Theorem 1.1 (Keimel, Werner [2]). *For every* $R\in\mathscr{V}$, $R\cong\mathrm{Adm}(\mathrm{Hom}_\sigma(R,\mathscr{K}),\mathscr{K})$.

In fact the correspondence between R and $\mathrm{Hom}_\sigma(R,\mathscr{K})$ gives rise to a duality between $\mathscr{V}$ and Boolean H-spaces.

The next lemma leads to the characterization of $F_{\mathscr{V}}(m)$, the free algebras in $\mathscr{V}$ with m generators. Let $\{A_i\colon i<n\}$ be the set of maximal members of $\mathscr{K}$ under inclusion.

Lemma 1.2. (a) *For* $R\in\mathscr{V}$, $\mathrm{Hom}_\sigma(R,\mathscr{K})=\bigcup_{i<n}\tilde{A}_i$.

(b) *If R is $\mathscr{V}$-free with a set I of generators,* $\tilde{A}_i\underset{\mathrm{top}}{=}A_i^I$ *(a product space).*

Proof. (a) Obvious since $\tilde{A}_i=\mathrm{Hom}_\sigma(R, A_i)$.

(b) Since R is free on I the restriction map $\rho\colon\tilde{A}_i\to A_i^I$ given by $\rho(\varphi)=\varphi|I$ for $\varphi\in\tilde{A}_i$ is a bijection between $\tilde{A}_i$ and A_i^I. Since A_i^I has a subbasis that consist of sets with the form $N_a^x=\{g\in A_i^I\colon g(x)=a\}$ for $x\in I$ and $a\in A_i$, it follows that ρ is continuous. It remains to show that

$$B=\{r\in R\colon\ \rho K(r;a)\ \text{is open in}\ A_i^I\ \text{for all}\ a\in A_i\}$$

is a subalgebra of R that contains the set of generators I. It is clear that $I\subseteq B$ so assume $r_1,\dots,r_k\in B$ and $\varphi\in K(f(r_1,\dots,r_k);a)$ for some operation f and $a\in A_i$. Since $\bigcap_{j=1}^{k}\rho K(r_j;\phi(r_j))$ is an open neighborhood of $\rho(\phi)=\phi|I$ it suffices to see that $\bigcap_{j=1}^{k}\rho K(r_j;\varphi(r_j))\subseteq\rho K(f(r_1,\dots,r_k);a)$. But this follows since R is free on I: for $g\in\rho K(r_j,\varphi(r_j))$ for all $j=1,\dots,k$, the unique $g^+\colon R\to A_i$ that extends g belongs to $K(r_j;\varphi(r_j))$ for all $j=1,\dots,k$ and hence

$$g^+(f(r_1,\dots,r_k))=f(g^+(r_1),\dots,g^+(r_k))=f(\varphi(r_1),\dots,\varphi(r_k))=\varphi f(r_1,\dots,r_k)=a,$$

i.e., $g\in\rho K(f(r_1,\dots,r_k))$ as desired.

Remark. For any set I and $A\in S(\mathscr{K})$, A^I is a closed subspace of $\bigcup_{i<n}A_i^I$. Each $\alpha\in H$ acts on $\bigcup_{i<n}A_i^I$ in the obvious way (for $\alpha\colon B\to C$ in H, $\bar{\alpha}\colon B^I\to C^I$ is given by $\bar{\alpha}(g)=\alpha\circ g$). For a free algebra R of $\mathscr{V}$ with a set I of generators it can be seen that $\mathrm{Hom}_\sigma(R,\mathscr{K})$ is isomorphic to $\bigcup_{i<n}A_i^I$ as Boolean H-spaces by comparing the actions of H on the two spaces.

For an infinite cardinal m the product space 2^m is the Stone space of a free Boolean algebra with m generators. These spaces are called Cantor m-spaces. It is easy to show that if S is a finite set with more than one element and $m \geqq \aleph_0$, then $S^m \underset{\text{top}}{=} 2^m$. Using this fact, Theorem 1.1 and Lemma 1.2 the free quasi-primal algebras are easy to describe.

Theorem 1.3. *Suppose $\mathscr{V}$ is a quasi-primal variety generated by a set $\mathscr{K}$ of finite algebras with n maximal members and suppose F is a free algebra of $\mathscr{V}$ with m generators. Then*

(i) *if $m \geqq \aleph_0$, $F \cong \mathrm{Adm}(X, \mathscr{K})$ where X is a union of n Cantor m-spaces.*

(ii) *if $m < \aleph_0$, $F \cong \mathrm{Adm}(X, \mathscr{K})$ where X is a discrete space with at most $|\mathscr{K}|^m$ elements.*

Theorem 1.3 has many special cases. We list a few.

Corollary 1.4. *Suppose F is a free algebra with m generators in a quasi-primal variety $\mathscr{V}$. Then*

(a) *if $m = \aleph_0$, $F \cong \mathrm{Adm}(2^{\aleph_0}, \mathscr{K})$,*

(b) *if $m \geqq \aleph_0$, and $\mathscr{V}$ is generated by a single quasi-primal algebra A, $F \cong \mathrm{Adm}(2^m, A)$. If, in addition, A is semi-primal (i.e., every $\alpha \in H$ extends to the identity), then F is a filtered Boolean power of A and if A is primal (i.e., no nontrivial subalgebras in addition to being semi-primal), then F is the Boolean power $A[B]$ where B is a free Boolean algebra with m generators.*

It follows from 1.3(ii) that a free algebra in $\mathscr{V}$ with a finite number of generators is finite and hence a direct product of the algebras in $\mathscr{K}$. The exact number of copies of each factor depends on the structure of the poset $S(\mathscr{K})$ as well as the number of isomorphisms between members of $S(\mathscr{K})$.

We illustrate how to use 1.3 with $m < \aleph_0$ by describing $F_{\mathscr{V}}(m)$ for the variety $\mathscr{V}_e'$ of all rings with unit that satisfy $x^{p^e} = x$ (p is a prime). Let GF_r denote the Galois field with p^r elements. Then $\mathscr{V}_e'$ is a semi-primal variety generated by GF_e. Let $F_e'(m)$ be the free algebra in $\mathscr{V}_e'$ with m generators. By a proof similar to that of 1.3, for all m,

$$F_e'(m) \cong \{f\colon (GF_e)^m \to GF_e\colon f \text{ is continuous and } f(GF_i^m) \subseteq GF_i \text{ for all } i < e\}.$$

Assume $m < \aleph_0$. Then $F_e'(m) \cong \prod_{g \in GF_e^m} B_g$ where $B_g \subseteq GF_e$ is to be determined. Partition $GF_e^m = GF_1^m \cup (GF_2^m \smallsetminus GF_1^m) \cup \cdots \cup (GF_e^m \smallsetminus GF_{e-1}^m)$ and observe that $f \in \mathrm{Adm}(GF_e^m, GF_e)$ $\Leftrightarrow f(GF_i^m \smallsetminus GF_{i-1}^m) \subseteq GF_i$ for all $i \leqq e$. Thus, $B_g = GF_i$ for each $g \in GF_i^m \smallsetminus GF_{i-1}^m$. Counting the number of times each GF_i is a factor we have

Corollary 1.5. *For $m < \aleph_0$, $F_e'(m) \cong \prod_{i=1}^{e} GF_i^{n_i}$ where $n_1 = p^m$ and for $1 < i \leqq e$, $n_i = (p^i)^m - (p^{i-1})^m$.*

It is easy to extend the above to the ring variety generated by GF_e where a unit is not required. The presence of the one element subalgebra of GF_1 means that GF_1 occurs as a factor only $p^m - 1$ times.

Section 2. Countable Chain Condition

All algebras considered in this section are assumed to have a constant 0 among the fundamental operations. Let $\mathrm{Si}(A)$ denote the class of subdirectly irreducible factors of A and, for a class $\mathscr{K}$ of algebras, let $\mathrm{Si}(\mathscr{K}) = \bigcup_{A \in \mathscr{K}} \mathrm{Si}(A)$.

For an algebra A a subset $D \subseteq A$ is called *separable* if for every $x, y \in D$, $x \neq y$ and every $\varphi \in \mathrm{Hom}(A, \mathrm{Si}(A))$ either $\varphi(x) = 0$ or $\varphi(y) = 0$. In other words, in every subdirectly irreducible factor of A at most one member of D has a nonzero value. We say that an algebra A satisfies the *countable separability condition* (c.s.c.) if there are no uncountable separable sets of nonzero elements of A.

The c.s.c. is intended to be a universal algebraic variant of the countable chain condition. We will see to what extent this is true after a useful characterization of separability is given.

For a algebra A the support of $a \in A$ is the set $\mathrm{Supp}(a) = \{\varphi \in \mathrm{Hom}(A, \mathrm{Si}(A)) : \varphi(a) \neq 0\}$. The following is obvious.

Lemma 2.1. *For $D \subseteq A$, D is separable iff every pair of distinct elements in D have disjoint supports.*

To begin the comparison of c.c.c. and c.s.c. we first notice that under fairly weak conditions c.c.c. implies c.s.c.

Lemma 2.2. *If $\mathrm{Si}(A) \models x \cdot 0 = 0 = 0 \cdot x$, then A satisfies c.s.c. whenever it satisfies c.c.c.*

Proof. By 2.1 it suffices to show that a pair of elements in A with disjoint supports must be disjoint. Suppose $\mathrm{Supp}(a) \cap \mathrm{Supp}(b) = \varnothing$ and $a \cdot b \neq 0$. There exist $\varphi \in \mathrm{Hom}(A, B)$ for some $B \in \mathrm{Si}(A)$ with $\varphi(a) \cdot \varphi(b) = \varphi(a \cdot b) \neq 0$. By the assumption on $\mathrm{Si}(A)$, $\varphi(a), \varphi(b) \neq 0$. Whence $\varphi \in \mathrm{Supp}(a) \cap \mathrm{Supp}(b)$, a contradiction. Thus, $a \cdot b = 0$ whenever a, b have disjoint supports.

The next result shows that c.c.c. and c.s.c. are equivalent in the situations that motivated this paper.

Theorem 2.3. *If $\mathrm{Si}(A) \models x \cdot y = 0 \leftrightarrow x = 0 \vee y = 0$, then A satisfies c.s.c. if and only if it satisfies c.c.c.*

Proof. The "if" part follows by 2.2. For the converse, it suffices to see that pairwise disjoint elements $a, b \in A$ have disjoint supports. For any $\varphi \in \mathrm{Hom}(A, \mathrm{Si}(A))$, $0 = \varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$; so the condition on $\mathrm{Si}(A)$ implies that either $\varphi \notin \mathrm{Supp}(a)$ or $\varphi \notin \mathrm{Supp}(b)$ as desired.

Corollary 2.4. *The c.c.c. and c.s.c. are equivalent for any Boolean algebra and for any member of the ring variety generated by a finite field. (Ring multiplication is the binary operation in the latter examples.)*

There are interesting situations not covered by 2.3. For example, cylindric and polyadic algebras with nonzero dimension. In these and other situations there is a meet semilattice reduct of the algebras so it is natural to investigate the equivalence of c.c.c. and c.s.c. for such algebras. Of course, 2.2 shows that c.c.c. implies c.s.c. In general the converse fails.

Remark 2.5. Every subdirectly irreducible algebra satisfies c.s.c. In fact, a maximal separable set of nonzero elements in a subdirectly irreducible algebra has one element. Thus, any simple monadic algebra with an uncountable number of atoms will satisfy c.s.c. but not c.c.c.

The next result establishes a partial converse to 2.2 for a class of varieties that includes all quasi-primal ones. As a preliminary, let $x \wedge y$ denote the binary operation that is defined on an algebra A by

$$x \wedge y = \begin{cases} x & \text{if } y \neq 0 \\ 0 & \text{if } y = 0. \end{cases}$$

By induction $x \wedge y$ can be extended to a n-ary function $\bigwedge_{i<n} x_i = (\ldots(x_0 \wedge x_1) \wedge \ldots \wedge x_{n-1})$ on A that has the property

(1) $A \models \bigwedge_{i<n} x_i = 0$ if and only if $x_i = 0$ for some $i < n$.

A class $\mathscr{K}$ of similar algebras is called *integral* if there is a polynomial in the fundamental operations that represents the function $x \wedge y$ simultaneously on each of the algebras in $\mathscr{K}$.

Theorem 2.6. *Suppose $\mathscr{K}$ is a finite set of finite algebras, $\mathscr{K}$ is integral, and $\mathrm{Si}(\mathscr{V}) \subseteq IS(\mathscr{K})$ where $\mathscr{V} = HSP(\mathscr{K})$. Furthermore, assume that the reduct $(A, \cdot, 0)$ is a meet semilattice with 0 whenever $A \in \mathscr{V}$. Then c.c.c. and c.s.c. are equivalent in $\mathscr{V}$.*

Proof. It is enough to assume $A \in \mathscr{V}$ satisfies c.s.c. and show that c.c.c. holds. Suppose D is an infinite set of pairwise disjoint nonzero elements in A. For $\varphi \in \mathrm{Hom}(A, \mathrm{Si}(A))$ let $D_\varphi = \{y \in D\colon \varphi(y) \neq 0\}$. Each D_φ is finite. (For if $x, y \in D$, $x \neq y$ and $\varphi(x) = \varphi(y)$, then $0 = x \cdot y$ implies $0 = \varphi(x) \cdot \varphi(y) = \varphi(x)$ by a semilattice property. Thus, φ can send at most one element of D to each nonzero element of $\varphi(A)$ which is finite.) Moreover, $\{|D_\varphi|\colon \varphi \in \mathrm{Hom}(A, \mathrm{Si}(A))\}$ is bounded by $\max\{|B|\colon B \in \mathrm{Si}(A)\}$. Because of this, for each $x \in D$ we can choose $D_{\varphi_x} \in \{D_\psi\colon \psi(x) \neq 0\}$ that has maximal size. The collection of finite sets $\{D_{\varphi_x}\colon x \in D\}$ forms a partially ordered set P under $\subseteq$. Choose a $D' \subseteq D$ such that $\{D_{\varphi_x}\colon x \in D'\}$ is the set of maximal elements of P (and no proper subset of D' will yield all maximal elements). From $D = \bigcup_{x \in D'} D_{\varphi_x}$ it follows that $|D'| = |D|$. Maximality in P gives

(2) $x, y \in D'$, $\quad x \neq y$ implies $D_{\varphi_x} \not\supseteq D_{\varphi_y}$.

For each $x \in D'$ let $n = |D_{\varphi_x}|$ and order $D_{\varphi_x} = \{a_i\colon i < n\}$. Then let $\bar{x} = \bigwedge_{i<n} a_i$. By (1), for each $x \in D'$,

(3) for each $\varphi \in \mathrm{Hom}(A, \mathrm{Si}(A))$, $\quad \varphi(\bar{x}) \neq 0$ iff $\varphi(y) \neq 0$ for all $y \in D_{\varphi_x}$.

In particular, (3) implies $\varphi_x(\bar{x}) \neq 0$; so $\bar{x} \neq 0$. Now suppose $x, y \in D'$, $x \neq y$ and $\psi(\bar{x}) \neq 0$ for some $\psi \in \mathrm{Hom}(A, \mathrm{Si}(A))$. By (3), $D_{\varphi_x} \subseteq D_\psi$; so $D_{\varphi_x} = D_\psi$ since D_{φ_x} was chosen with maximal size. Thus, by (2), there exist $z \in D_{\varphi_y} \setminus D_\psi$. Hence $\psi(z) = 0$ and (3) yields $\psi(\bar{y}) = 0$. It follows that for $x, y \in D'$, $\bar{x}$ and $\bar{y}$ have disjoint supports.

(Hence, $\bar{x} \neq \bar{y}$ also.) Thus $\{\bar{x}: x \in D'\}$ is a separable set of nonzero elements of A. Assuming c.s.c. holds in A, $|D| = |D'| \leqq \aleph_0$ as desired.

Corollary 2.7. *If $\mathscr{V}$ is a quasi-primal variety and the reduct $(A, \cdot, 0)$ of every $A \in \mathscr{V}$ is a meet semilattice with 0, then c.c.c. and c.s.c. are equivalent in $\mathscr{V}$.*

Proof. $\mathrm{Si}(\mathscr{V})$ is integral since $x \wedge y = d(x, d(x, y, 0), 0)$ where $d(x, y, z)$ is the ternary discriminator

$$d(x, y, z) = \begin{cases} x & \text{if } y = z \\ z & \text{if } y \neq z. \end{cases}$$

We now consider the question raised in the introduction and answer it affirmatively for c.s.c.

A topological space X has the countable chain condition (c.c.c.) if every collection of pairwise disjoint nonempty open subsets of X is countable. An obvious cardinality argument gives the following.

Lemma 2.8. *If a space Y is a countable union of closed subspaces $X_i (i < \omega)$ and each X_i has the c.c.c., then Y has the c.c.c.*

Theorem 2.9. *Every free algebra F in a quasi-primal variety $\mathscr{V}$ satisfies c.s.c.*

Proof. We may assume F has $m \geqq \aleph_0$ generators. Using $\mathscr{K} = \mathrm{Si}(\mathscr{V})$ in Theorem 1.3(i), $F \cong \mathrm{Adm}(X, \mathscr{K})$ where $X = \mathrm{Hom}_\sigma(F, \mathscr{K})$ is a finite union of Cantor m-spaces. Thus, it follows from 2.8 that the space X has the c.c.c. since the spaces 2^m have c.c.c. Now suppose $\{A_i: i < n\}$ are the maximal members of $\mathrm{Si}(\mathscr{V})$ and $r \in F$. Then

$$\mathrm{Supp}(r) = \bigcup_{i < n} \bigcup_{0 \neq a \in A_i} K(r; a)$$

is a clopen subset of X. By Lemma 2.1, the c.s.c. for F follows from the c.c.c. for X.

As a corollary to 2.9 and 2.4 it follows that the c.c.c. holds in every free algebra in the ring variety generated by a finite field. As a corollary to 2.9 and 2.7 the c.c.c. holds in every free algebra of a quasi-primal variety of cylindric algebras. Many other special cases can be enunciated by consulting a list of quasi-primal varieties (c.f., Werner [3]).

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