Math. Japonica 25, No. 1 (1980), 111–115.

THE REPRESENTATION OF IMPLICATIVE *BCK*-ALGEBRAS

Stephen D. COMER*

(Received September 27, 1979)

Abstract. It is shown that the only subdirectly irreducible implicative *BCK*-algebra is the standard 2-element one. Consequently, that fragment of the calculus of classes which involves only the set difference operation is completely axiomatized by the axioms for implicative *BCK*-algebras.

Y. Imai and K. Iséki introduced a class of algebras in [2] to serve as models for the implicative sentential calculus. Today these algebras are known as implicative *BCK*-algebras. Apparently set-theoretic considerations also partly motivated the definition of these algebras. In the survey article [4], Iseki and Tanaka mention that *BCK*-algebras generalize properties of the set difference operation.

Problem. Axiomatize those properties of the calculus of classes that only involve the set difference operation (and the empty set).

It is easily verified that every collection of sets closed under set difference is an implicative *BCK*-algebra. The goal of this note is to show that, conversely, every implicative *BCK*-algebra is, up to isomorphism, a set difference algebra. Thus, the axioms for implicative *BCK*-algebras completely characterize properties of set difference.

The main result reduces to showing that the only subdirectly irreducible implicative *BCK*-algebra is the 2-element one. The proof utilizes a characterization of maximal ideals in implicative *BCK*-algebras that generalizes facts about Boolean algebras. For unexplained facts about *BCK*-algebras the reader should see [4].

1. Preliminaries. For any set X suppose $\mathscr{S}$ is a collection of subsets of X such that (i) $\emptyset \in \mathscr{S}$ and (ii) the set difference $A-B \in \mathscr{S}$ whenever $A, B \in \mathscr{S}$. The system $(\mathscr{S}, -, \emptyset)$ is called a *set difference algebra.* From the viewpoint of universal algebra $(\mathscr{S}, -, \emptyset)$ is a subalgebras of the system $(P(X), -, \emptyset)$.

A system $(A, *, 0)$ where A is a nonempty set, $0 \in A$, and $*$ is a binary operation on A is an *implicative BCK-algebra*, *IBCK-algebra* for short, if it satisfies the following axioms:

(I) $(x*y)*(x*z) \leqslant z*y.$

AMS(MOS) subject classifications. 03G10, 06A12.

* The Citadel, Charleston, S.C. 29409, U,S.A.

(II) $x * (x * y) \leqslant y$.

(III) $x \leqslant x$.

(IV) $0 \leqslant x$.

(V) $x \leqslant y$ and $y \leqslant x$ imply $x = y$.

(VI) $x \leqslant y$ if and only if $x * y = 0$.

(VII) $x * (y * x) = x$.

Models of (I) – (VI) are called *BCK-algebras.* The following lemma presents properties derived in [4] from the axioms above.

Lemma 1. *In an IBCK-algebra, the following hold:*

(1) $(x * y) * z = (x * z) * y$.

(2) $x * y \leqslant z$ *implies* $x * z \leqslant y$.

(3) $x \leqslant y$ *implies* $x * z \leqslant y * z$ *and* $z * y \leqslant z * x$.

(4) $y * x \leqslant y$.

(5) $x * 0 = x$.

(6) $x * (x * y) = y * (y * x)$.

(7) $y * x = (y * x) * x$.

Properties (1) – (5) hold in any *BCK*-algebra. A *BCK*-algebra that satiafies (6) is called *commutative.* A binary operation $\wedge$ can be defined in any *BCK* algebra by

$$x \wedge y = x * (x * y).$$

Since an *IBCK*-algebra is commutative, $(A, \wedge)$ is a semilattice where $x \wedge y$ is the greatest lower bound of x and y.

H. Yutani proved in [5] that the class of commutative *BCK*-algebras form a variety. Consequently the class of all *IBCK*-algebra is also a variety.

A nonempty subset I of a *BCK*-algebra A is called an *ideal* if (i) $0 \in I$ and (ii) $y * x$, $x \in I$ implies $y \in I$. An ideal I of A is *maximal* if $I \neq A$ and there is no ideal J such that $I < J < A$. It is an easy consequence of the definition that $y \leqslant x \in I$ implies $y \in I$.

As in the case of Boolean algebras, ideals correspond in a one-one way with homomorphisms. See [4] for details.

2. Ideals in *BCK*-algebras. Given a subset X of a *BCK*-algebra A let $I_g(X)$ denote the smallest ideal of A that contains X. In particular, $\{0\} = I_g(\emptyset)$ is the smallest ideal in a *BCK*-algebra and is called the *trivial* ideal.

Our first result gives a generative description of ideals. See also Iséki and Tanaka [3].

Proposition 2. *For any nonempty subset X in a BCK-algebra A, $y \in I_g(X)$ if and only if*

(8) *There exists a finite sequence $x_1, \cdots, x_r \in X$ such that*

$$(\cdots((y * x_1) * x_2) * \cdots) * x_r = 0.$$

Corollary 3. *In an IBCK-algebra A the principal ideal generated by x is* $I_g(\{x\}) = \{y \in A : y \leqslant x\}$.

Proof. If y satisfies (8), then $(\cdots(y * x) * \cdots) * x = 0$. By (7) this condition is equivalent to $y * x = 0$, that is, $y \leqslant x$.

In the proof of Theorem 5 we need a generalization of the above corollary. The next result describes principal extentions of ideals in *IBCK*-algebras.

Corollary 4. *For any ideal I in an IBCK-algebra A, $y \in I_g(I \cup \{x\})$ if and only if*

(9) $y \in I$ *or* $y * x \in I$.

Proof. Clearly (9) implies (8). Assume that y satisfies (8). Then $(\cdots(y * x_1) * \cdots) * x_r = 0$ where for each i, $x_i \in I$ or $x_i = x$. If $x_i \notin I$ for all i, then $y \in I_g(\{x\})$; hence y satisfies (9) since $y * x = 0 \in I$ by Corollary 3. Now suppose that $x_i \in I$ for some i. By repeated use of (1) we may rearrange $x_1, \cdots, x_r$ and assume without loss of generality that

$$(\cdots(((\cdots(y * x'_1) * \cdots) * x'_n) * x'_{n+1}) * \cdots) * x'_r = 0 \in I$$

where $x'_i \in I$ for all $i = n+1, \cdots, r$ and $x'_i = x$ for $i \leqslant n$ (possibly $n = 0$ in which case $x_i \in I$ for all i). Repeatedly using condition (ii) in the definition of ideal we obtain either $y \in I$ (when $n = 0$) or $(\cdots(y * x) * \cdots) * x \in I$. By (7) the last alternative is equivalent to $y * x \in I$. Hence y satisfies (9).

Both corollaries 3 and 4 extend to *BCK*-algebras by omitting the redution from $(\cdots(y * x) * \cdots) * x$ to $y * x$. For example, in a *BCK*-algebra A the principal ideal generated by an element x is $\{y \in A : (\cdots(y * x) * \cdots) * x = 0$ where x is repeated a finite number of times$\}$. There results are not needed so we will not state them.

The following characterizations of maximal ideals in *IBCK*-algebras extend results known about Boolean algebras. It would be of interest to obtain similar characterizations for maximal ideals in commutative *BCK*-algebras.

Theorem 5. *For an ideal I in an IBCK-algebra A, the following conditions are equivalent:*

(10) $I \neq A$, *and for any* $x, y \in A$ *either* $x * y \in I$ *or* $y * x \in I$.

(11) $I \neq A$, *and for any* $x, y \in A$, *if* $x \wedge y \in I$, *then* $x \in I$ *or* $y \in I$.

(12) I *is a maximal ideal.*

Proof. (10) ⇒ (11): Assume (10) and consider $x, y \in A$ with $x \wedge y \in I$. By (10) either $x * y \in I$ or $y * x \in I$. If $x * y \in I$, then it follows from $x * (x * y) = x \wedge y \in I$ that $x \in I$. The proof that $y * x \in I$ implies $y \in I$ is similar.

(11) ⇒ (12): Suppose (11) and that J is an ideal where $I < J$. Thus, there exists an $x \in J - I$. Suppose $y \in A$ is arbitrary. By (VII) and (III), $x \wedge (y * x) = x * (x * (y * x)) = x * x = 0 \in I$. Since $x \notin I$ by assumption, (11) implies that $y * x \in I \subset J$. Since $x \in J$ and J is an ideal, $y \in J$. Thus, $J = A$.

(12) ⇒ (10): Assume I is maximal and suppose $x * y \notin I$. Then, by maximality, $I_g(I \cup \{x * y\}) = A$. Hence $y \in I_g(I \cup \{x * y\})$ and consequently, by Corollary 4, either $y \in I$ or $y * (x * y) \in I$. By (VII) $y \in I$ in both cases and thus $y * x \in I$ since $y * x \leqslant y$. Hence $y * x \in I$ whenever $x * y \in I$.

3. Representation. A *BCK*-algebra A is *subdirectly irreducible* if $|A| > 1$ and A contains a smallest nontrivial ideal. The second condition is equivalent to the ideal $\{0\}$ being *meet-irreducible*, that is, for all ideals I, J, if $\{0\} = I \cap J$, then $\{0\} = I$ or $\{0\} = J$. A *BCK*-algebra is *simple* if $\{0\}$ is a maximal ideal.

Let **2** denote the 2-element implicative *BCK*-algebra $(\{0, 1\}, *, 0)$ where $1 * 0 = 1$ and $0 * 0 = 0 * 1 = 1 * 1 = 0$. **2** is isomorphic to the set difference algebra of all subsets of a 1-element set.

Lemma 6. *Every subdirectly irreducible IBCK-algebra is isomorphic to* **2**.

Proof. Suppose A is a subdirectly irreducible *IBCK*-algebra. We use (11) in Theorem 5 to show $\{0\}$ is maximal. Suppose $x \wedge y = 0$. Then, using Corollary 3, $I_g(\{x\}) \cap I_g(\{y\}) = I_g(\{x \wedge y\}) = \{0\}$ so either $I_g(\{x\}) = \{0\}$ or $I_g(\{y\}) = \{0\}$ since $\{0\}$ is meet-irreducible. Thus, $x = 0$ or $y = 0$ which shows that $\{0\}$ satisfies (11). By (10) in Theorem 5, for every $x, y \in A$ either $x \leqslant y$ or $y \leqslant x$. That is, A is linearly ordered by $\leqslant$. If $|A| > 2$, say $0 < a < b$, then $I_g(\{a\})$ is a proper, nontrivial ideal of A contradicting the maximality of $\{0\}$. Thus, $|A| = 2$ and, therefore, is isomorphic to **2**.

Theorem 7. *Every implicative BCK-algebra is isomorphic to a set difference* *algebra*

Proof. The class of *IBCK*-algebras form a variety. By a theorem of Birkhoff (cf., Grätzer [1], section 20), every *IBCK*-algebra A is isomorphic to a subdirect product of subdirectly irreducible *IBCK*-algebra. By Lemma 6 this means that A is embeddable in a power 2^X for some set X. Since $2^X \cong (P(X), -, \emptyset)$ the result follows.

Referen

References

References

[1] G. Grätzer: Universal Algebra, D. Von Nostrand Co., Inc., Princeton, 1968.

[2] Y. Imai and K. Iséki: On axiom systems of propositional calculi XIV, Proc. Japan Acad., 42 (1966), 19–22.

[3] K. Iséki and S. Tanaka: Ideal theory of *BCK*-algebras, Math. Japon., 21 (1976), 351–366.

[4] K. Iséki and S. Tanaka: An introduction to the theory of *BCK*-algebras, Math. Japon., 23 (1978), 1–26.

[5] H. Yutani: The class of commutative *BCK*-algebras is equationally definable, Math. Sem. Notes Kobe Univ., 5 (1977), 207–210.