Zeitschr. f. math. Logik und Grundlagen d. Math.
Bd. 27, S. 557–560 (1981)

THE DECISION PROBLEM FOR CERTAIN NILPOTENT CLOSED VARIETIES

by STEPHEN D. COMER in Charleston, South Carolina (U.S.A.)

1. Preliminaries

The nilpotent closure operation $J_{\mathcal{N}}$ on the lattice of varieties was introduced by I. I. MEL'NIK in [5]. This note contains some observations about how this operation effects the decidability of a variety. We show that if $\mathcal{V}_0$ and $\mathcal{V}_1$ are independent varieties the nilpotent closure of $\mathcal{V}_0 + \mathcal{V}_1$ has an inseparable first-order theory. At the other extreme we show the nilpotent closure of a variety of projection algebras has a decidable first-order theory.

The author wishes to thank B. M. SCHEIN for bringing MEL'NIK's work to his attention.

We consider algebras $\mathfrak{A} = (A, F_i^{\mathfrak{A}}(i \in I))$ with a fixed similarity type $\tau = (n_i : i \in I)$ where $n_i \geqq 1$ for all $i \in I$ (i e., no 0-ary operations are allowed).

Several special varieties will appear below. Let $\mathcal{N}$ denote the variety defined by the set of equations

$$\{F_i(x_1, \ldots, x_n) = F_j(y_1, \ldots, y_m) : i, j \in I\}.$$

$\mathcal{N}$ is called the (τ)-*nilpotent variety*; its members are clearly "constant" algebras. For each function σ from I into the positive integers such that $\sigma_i \leqq n_i$ for all i we let $\mathcal{P}_\sigma$ denote the variety defined by the equations

$$\{F_i(x_1, \ldots, x_n) = x_{\sigma(i)} : i \in I\}.$$

The members of $\mathcal{P}_\sigma$ are called (σ)-*projection algebras*. Obviously $\mathcal{N}$ and $\mathcal{P}_\sigma$ are atoms in the lattice L_τ of all varieties of τ-algebras. The varieties are categorical in every power and hence decidable varieties (i.e., have a decidable first-order theory).

The operation $J_{\mathcal{N}}$ is defined for a variety $\mathcal{V}$ by $J_{\mathcal{N}}(\mathcal{V}) = \mathcal{V} + \mathcal{N}$, the sum of $\mathcal{V}$ and $\mathcal{N}$ in the lattice L_τ. Since $J_{\mathcal{N}}(\mathcal{V}) = \mathcal{V}$ whenever $\mathcal{V} \supseteq \mathcal{N}$, we assume $\mathcal{V} \not\supseteq \mathcal{N}$. This operation, called the *nilpotent closure*, was introduced in [5]. The following lemma is from that paper.

LEMMA 1. (MEL'NIK) *Suppose $\mathcal{V}$ is a variety and $\mathcal{V} \not\supseteq \mathcal{N}$. There exist a unary term t (in the language of $\mathcal{V}$) such that $\mathcal{V}$ has a equational bases $\{t(x) = x\} \cup \Sigma_{\mathcal{V}}^{\mathcal{N}}$ where $\Sigma_{\mathcal{V}}^{\mathcal{N}}$ is the set of all equations that hold in both $\mathcal{V}$ and $\mathcal{N}$ (i.e., equations $t_1 = t_2$ that hold in $\mathcal{V}$ and neither t_1 nor t_2 are variables).*

Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are algebras with similarity type τ. $\mathfrak{B}$ is called an *N-extension* of $\mathfrak{A}$ (see [5]) if (i) $\mathfrak{A}$ is isomorphic to the algebra $\Omega(\mathfrak{B}) = \bigcup\{F_i^{\mathfrak{B}}(b_1, \ldots, b_n) : i \in I$ and $b_1, \ldots, b_n \in B\}$; and (ii) $\Omega(\mathfrak{B})$ is a retract of $\mathfrak{B}$ (i.e., there exist an endomorphism of $\mathfrak{B}$ that is the identity on $\Omega(\mathfrak{B})$).

Suppose X is a family of mutually disjoint sets indexed by A such that $a \in X_a$ for all $a \in A$. We form an extension $\mathfrak{A}[X]$ of the algebra $\mathfrak{A}$ on the universe $A[X] = \bigcup\{X_a : a \in A\}$. For each $i \in I$, we define $F_i^{\mathfrak{A}[X]}(x_1, \ldots, x_n) = F_i^{\mathfrak{A}}(a_1, \ldots, a_n)$ where $x_i \in X_{a_i}$

for $i = 1, \ldots, n$. $\mathfrak{A}[X]$ is called a *inflation* of $\mathfrak{A}$. For semigroups this notion was introduced in [1], p. 98. The following characterizes the algebras in $J_{\mathcal{N}}(\mathcal{V})$.

Proposition 2. *The following are equivalent for any variety* $\mathcal{V}$*:*

(1) $\mathfrak{B} \in J_{\mathcal{N}}(\mathcal{V})$;

(2) $\mathfrak{B}$ *is an* $\mathcal{N}$*-extension of some* $\mathfrak{A} \in \mathcal{V}$*;*

(3) $\mathfrak{B}$ *is isomorphic to an inflation of some* $\mathfrak{A} \in \mathcal{V}$.

Proof. See [5] for the equivalence of (1) and (2). (2) implies (3). Let $\mathfrak{A} = \Omega(\mathfrak{B})$ and, for each $a \in \Omega(\mathfrak{B})$, let $X_a = \{b \in B : \varphi(b) = a\}$ where φ is the retract of $\mathfrak{B}$ onto $\Omega(\mathfrak{B})$. (3) implies (1). Suppose $\mathfrak{B} = A[X]$. The map $\varphi : \mathfrak{B} \to \mathfrak{A}$ defined by $\varphi(y) = a$ if $y \in X_a$ is a retract onto $\mathfrak{A} \in \mathcal{V}$. $\theta = (A \times A) \cup I_{\mathfrak{B}}$ is a congruence relation on $\mathfrak{B}$ whose quotient $\mathfrak{B}/\theta$ belongs to $\mathcal{N}$. $\theta \cap \ker(\varphi) = I_{\mathfrak{B}}$ so $\mathfrak{B}$ belongs to $\mathcal{V} + \mathcal{N}$.

As an immediate consequence of Lemma 1 and Proposition 2 we obtain:

Corollary 3. *An equation holds in* $J_{\mathcal{N}}(\mathcal{V})$ *iff it is derivable from* $\Sigma_{\mathcal{V}}^{\mathcal{N}}$.

Two varieties $\mathcal{V}_0$ and $\mathcal{V}_1$ in L_τ are called *independent* if there exist a binary term $b(x, y)$ in the language of algebras of type τ such that $b(x, y) = x$ holds in $\mathcal{V}_0$ and $b(x, y) = y$ holds in $\mathcal{V}_1$. This notion was introduced by Foster [4].

2. An Undecidability Result

Recall that a first-order theory is *inseparable* if there is no recursive set separating the logically valid sentences of the language from the sentences that fail in some model of the theory. This is a strong form of undecidability; it implies the theory is hereditarily undecidable (i.e., every subtheory is also undecidable). The main technique for showing that a theory T is inseparable involves interpreting into T a theory already known to be inseparable. For a detailed description of this method see [7] or [6].

Theorem 4. *Suppose* $\mathcal{V}_0$ *and* $\mathcal{V}_1$ *are independent varieties and* $\mathcal{V} = \mathcal{V}_0 + \mathcal{V}_1$ *does not contain* $\mathcal{N}$. *Then* $J_{\mathcal{N}}(\mathcal{V})$ *is inseparable (and hence hereditarily undecidable).*

Proof. Suppose $b(x, y)$ is the binary term that shows $\mathcal{V}_0$ and $\mathcal{V}_1$ are independent and suppose $t(x)$ is the unary term (from Lemma 1) such that $t(x) = x$ holds in $\mathcal{V}$ but not in $\mathcal{N}$. We show $J_{\mathcal{N}}(\mathcal{V})$ is inseparable by interpreting the theory DE_2 of two disjoint equivalence relations into it. This theory is known to be inseparable (cf., [2] Theorem 3 or [6], Theorem 16.56).

The interpretation is given by the formulas

$$D(x) := \forall y(t(y) = x \to y = x),$$
$$E(x, y) := \forall z(b(z, x) = b(z, y)),$$
$$F(x, y) := \forall z(b(x, z) = b(y, z)).$$

By Theorem 1 in [2] (or Proposition 15.17 in [6]) it suffices to verify

$$(*) \quad \begin{cases} \text{for every finite model } (A, R, S) \text{ of } \mathrm{DE}_2 \text{ there exist } \mathfrak{B} \text{ in } J_{\mathcal{N}}(\mathcal{V}) \\ \text{such that } (A, R, S) \cong (D^{\mathfrak{B}}, E^{\mathfrak{B}}, F^{\mathfrak{B}}). \end{cases}$$

Given (A, R, S) let $\{R_0, \ldots, R_{n-1}\}$ denote the R-equivalence classes and $\{S_0, \ldots, S_{m-1}\}$ denote the S-equivalence classes of A. Choose $\mathfrak{B}_i \in \mathscr{V}_i$ and a one-one function f_i $(i = 0, 1)$ such that $f_0 \colon m \to \mathfrak{B}_0$ and $f_1 \colon n \to \mathfrak{B}_1$. Since R and S are disjoint equivalence relations on A, $S_i \cap R_j$ contains at most one element for each $(i, j) \in m \times n$. Thus, $f_0 \times f_1$ sets up a natural bijection between A and the subset

$$\bar{A} = \{(f_0(i), f_1(j)) \colon S_i \cap R_j \neq 0\}$$

of $B_0 \times B_1$. By adding elements to each element in $B_0 \times B_1 \setminus \bar{A}$ we obtain an inflation $\mathfrak{B} = (\mathfrak{B}_0 \times \mathfrak{B}_1)[X]$ of $\mathfrak{B}_0 \times \mathfrak{B}_1$ that satisfies (*).

Remarks. 1. If $\mathscr{V}_0$ and $\mathscr{V}_1$ have arbitrarily large finite models then $\mathfrak{B}$ may be chosen to be finite. In this case the first-order theory of $J_{\mathscr{N}}(\mathscr{V})$ is finitely inseparable. 2. It follows from Corollary 3 *that* $J_{\mathscr{N}}(\mathscr{V})$ *has a decidable equational theory whenever* $\mathscr{V}$ *does.* In section 4 we mention independent varieties $\mathscr{V}_0$ and $\mathscr{V}_1$ such that $\mathscr{V} = \mathscr{V}_0 + \mathscr{V}_1$ is decidable. By Theorem 4 $J_{\mathscr{N}}(\mathscr{V})$ is undecidable. Thus $J_{\mathscr{N}}$ does not preserve first-order decidability.

3. A Decidability Result

We show that $J_{\mathscr{N}}(\mathscr{V})$ is a decidable variety whenever $\mathscr{V}$ is a variety of projection algebras. The proof uses the idea of m-elementary subsystem, $\mathfrak{A} \leqq_m \mathfrak{B}$, introduced by Dana Scott (cf. [6], pp. 352ff.) and closely resembles the proof that the theory of one equivalence relation is decidable.

Suppose $\mathscr{P}_\sigma$ is a variety of projection algebras. By Proposition 2 every member of $J_{\mathscr{N}}(\mathscr{P}_\sigma)$ is an inflation $\mathfrak{A}[X]$ of an algebra $\mathfrak{A}$ in $\mathscr{P}_\sigma$. For each $m < \omega$, we say that $\mathfrak{A}[X]$ in $J_{\mathscr{N}}(\mathscr{P}_\sigma)$ is *m-basic* if

(i) $|X_a| \leqq m + 1$ for every $a \in A$;

(ii) for every $n > 0$, $|\{a \in A \colon |X_a| = n\}| \leqq m$.

Theorem 5. *$J_{\mathscr{N}}(\mathscr{P}_\sigma)$ is a decidable variety (i.e., has a decidable first-order theory).*

Proof. A straightforward counting argument shows that every m-basic $\mathfrak{A}[X]$ in $J_{\mathscr{N}}(\mathscr{P}_\sigma)$ has at most $\dfrac{m(m+1)(m+2)}{2}$ elements and there are at most $(m+1)^{m+1} - 1$ isomorphism types of m-basic algebras in $J_{\mathscr{N}}(\mathscr{P}_\sigma)$. Thus it can be decided whether or not a sentence holds in every m-basic algebra. Hence it suffices to show that every $\mathfrak{B}[Z]$ in $J_{\mathscr{N}}(\mathscr{P}_\sigma)$ contains an m-basic, m-elementary subsystem (see [6], p. 352). This is done in two steps.

(1) If $|Z_b| \geqq m + 1$, choose $Y_b \subseteqq Z_b$ with $b \in Y_b$ and $|Y_b| = m + 1$; otherwise let $Y_b = Z_b$. This yields a $\mathfrak{B}[Y] \leqq_m \mathfrak{B}[Z]$ where $|Y_b| \leqq m + 1$ for all $b \in B$.

(2) For each positive integer k, let $B_k = \{b \in B \colon |Y_b| = k\}$. The collection $\{B_k \colon k = 1, \ldots, m + 1\}$ partitions B. For each k, set $A_k = B_k$ if $|B_k| \leqq m$ and let A_k be a fixed subset of B_k with $|A_k| = m$ in case $|B_k| > m$. Then $A = \bigcup\{A_k \colon k = 1, \ldots, m + 1\}$ becomes a subalgebra of $\mathfrak{B}$. Defining X as the restriction of Y to A we obtain an m-basic algebra $\mathfrak{A}[X]$ and $\mathfrak{A}[X] \leqq_m \mathfrak{B}[Y]$.

The desired conclusion follows from (1) and (2).

4. Applications to Semigroups

We look at what theorems 4 and 5 mean in the case of groupoids. For algebras where $x \cdot y$ is the fundamental operation there are only two possibilities for $\mathscr{P}_\sigma$: the variety $\mathscr{L}$ of all left zero semigroups (defined by $xy = x$) and the variety $\mathscr{R}$ of all right zero semigroups (defined by $xy = y$). The variety $\mathscr{N}$ (defined by $xy = uv$) consist of all constant semigroups. The variety $\mathscr{L}^+ = J_{\mathscr{N}}(\mathscr{L})$ (respectively, $\mathscr{R}^+ = J_{\mathscr{N}}(\mathscr{R})$) is defined by the laws $(xu)(vy) = xy$ and $xy = xz$ (respectively, $(xu)(vy) = xy$ and $xy = zy$).

Both $\mathscr{L}^+$ and $\mathscr{R}^+$ are decidable by Theorem 5. The varietal product $\mathscr{L} \otimes \mathscr{R}$ (introduced by WALTER TAYLOR [8]) is the variety of all rectangular bands (defined by $(xu)(vy) = xy$ and $x^2 = x$). It is an immediate consequence of [3] that $\mathscr{L} \otimes \mathscr{R}$ is a decidable variety. Theorem 4 shows that the nilpotent closure $J_{\mathscr{N}}(\mathscr{L} \otimes \mathscr{R})$ is hereditarily undecidable even though every proper subvariety is decidable.

The operation $J_{\mathscr{N}}$ is an endomorphism of L_τ so $J_{\mathscr{N}}(\mathscr{L} \otimes \mathscr{R}) = \mathscr{L}^+ + \mathscr{R}^+$; however, it cannot be a varietal product of $\mathscr{L}^+$ and $\mathscr{R}^+$ (for then it would be decidable).

References

[1] CLIFFORD, A. H., and G. B. PRESTON, The algebraic theory of semigroups, Vol. I. Amer. Math. Soc. 1961.
[2] COMER, S. D., Finite inseparability of some theories of cylindrification algebras. J. Symbolic Logic **34** (1969), 171–176.
[3] FEFERMAN, S., and R. L. Vaught, The first order properties of products of algebraic systems. Fund. Math. **47** (1959), 57–103.
[4] FOSTER, A. L., The identities of — and unique subdirect factorization within — classes of universal algebras. Proc. AMS **7** (1956), 1011–1013.
[5] MEL'NIK, I. I., Nilpotent shifts on manifolds. Mat. Zametki **14** (1973), 703–712. Translated: Math. Notes **14** (1973), 962–966.
[6] MONK, J. D., Mathematical Logic. Springer-Verlag, Berlin–Heidelberg–New York 1976.
[7] RABIN, M. O., A simple method for undecidability proofs and some applications. In: Logic, Methodology and Philos. Sci. (Y. BAR HILLEL, editor). North-Holland Publ. Comp., Amsterdam 1965, pp. 58–68.
[8] TAYLOR, W., Characterizating Mal'cev condition. Algebra Univ. **3** (1973), 351–397.

(Eingegangen am 20. März 1980)