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#21

EXTENSION OF POLYGROUPS BY POLYGROUPS AND THEIR REPRESENTATIONS USING COLOR SCHEMES

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In this paper we introduce a construction for building a "big" polygroup from two "small" ones and show that the important classes of polygroups are closed under this construction. In general, it is very hard to determine whether a given polygroup is chromatic or not. A sufficient condition was given in Section 5 of [2] and generalized in [3]. The construction here gives an easy way to show that a large number of polygroups are chromatic. In view of Theorem 5 perhaps the product $\mathfrak{A}[\mathfrak{B}]$ could be called the *wreath product* of $\mathfrak{A}$ by $\mathfrak{B}$.

To make the paper reasonably self-contained basic definitions are collected in Section 1. The product construction $\mathfrak{A}[\mathfrak{B}]$ is described in Section 2. Section 3 contains the main results, namely, that the product operation $\mathfrak{A}[\mathfrak{B}]$ preserves various properties. For example Theorems 2 and 6 show that polygroups $\mathfrak{A}$ and $\mathfrak{B}$ are chromatic iff $\mathfrak{A}[\mathfrak{B}]$ is chromatic. An easy application of the product construction to the study of relation algebras is given in Section 4.

1. PRELIMINARIES.

We recall a few basic definitions from [2].

A *polygroup* is a system $\mathfrak{M} = \langle M,\cdot,e,^{-1}\rangle$ where $e \in M$, $^{-1}$ is a unary operation on M, $\cdot$ maps M^2 into nonempty subsets of M, and the following axioms hold for all $x,y,z \in M$:

(P_1) $(x\cdot y)\cdot z = x\cdot(y\cdot z)$

(P_2) $x\cdot e = x = e\cdot x$

(P_3) $x \in y\cdot z$ implies $y \in x\cdot z^{-1}$ and $z \in y^{-1}\cdot x$.

Many important polygroups are derived from color schemes, a notion that extends D.G. Higman's homogeneous coherent configuration (see [5]). Suppose C is a set (of colors) and ι is an involution of C. A *color scheme* is a system $\mathcal{V} = \langle V, C_a\rangle_{a\in C}$ where

(1) $\{C_a : a \in C\}$ is a partition of $V^2 - Id = \{(x,y) \in V^2 : x \neq y\}$,

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(ii) $C_a^{\smile} = C_{\iota(a)}$ for all $a \in C$,

(iii) for each color and vertex the color is present on some edge from the vertex,

(iv) for $a,b,c \in C$ if $(x,y) \in C_c$ the existence of an (a,b)-path from x to y is independent of the choice of x and y, in symbols,

$$C_c \cap (C_a | C_b) \neq \emptyset \quad \Rightarrow \quad C_c \subseteq C_a | C_b .$$

Given a color scheme $\mathscr{V}$, choose a new symbol $I \notin C$. (Think of I as the identity relation.) The *algebra* (*color algebra*, or *configuration algebra*) of $\mathscr{V}$ is the system

$$\mathfrak{M}_{\mathscr{V}} = \langle C \cup \{I\}, \cdot, I, ^{-1} \rangle$$

where $a^{-1} = \iota(a)$ for $a \in C$, $I^{-1} = I$, $x \cdot I = x = I \cdot x$ for all $x \in C \cup \{I\}$, and for a,b, and c in C,

$$a \cdot b = \{c \in C : C_c \subseteq C_a | C_b \} \cup \{I : b = a^{-1}\}.$$

A polygroup is *chromatic* if it is isomorphic to the algebra of some color scheme.

A natural example of a chromatic polygroup is the system $G/\!/H$ of all double cosets of a group modulo a subgroup H. Namely,

$$G/\!/H = \langle \{HgH : g \in G\}, \cdot, H, ^{-1} \rangle$$

where $(HgH) \cdot (Hg'H) = \{Hghg'H : h \in H\}$ and $(HgH)^{-1} = Hg^{-1}H$. That these systems are chromatic was established in [2]. The double coset construction generalizes to the idea of a double quotient. This idea will not be needed in this paper for a general polygroup but only for ordinary groups. The general notion (see [2]) is equivalent to the following when restricted to groups. An equivalence relation θ on a group G is called a *conjugation* on G if

(i) $(\theta x)^{-1} = \theta x^{-1}$ for all x, and

(ii) $\theta(xy) \subseteq (\theta x)(\theta y)$ for all $x,y \in G$.

The natural quotient system $G/\!/\theta$ is a chromatic polygroup ([2]) and we define

$$Q^2(\text{Group}) = \{ G/\!/\theta : \theta \text{ is a conjugation on some group } G\}.$$

A conjugation θ is called *special* if it satisfies

(iii) $x\theta e \quad \Rightarrow \quad x = e.$

The class of all polygroups isomorphic to double quotients of groups via special conjugations is denoted by $Q_s^2(\text{Group})$.

2. AN EXTENSION CONSTRUCTION. Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are polygroups whose elements have been renamed so that $A \cap B = \{e\}$ where e is the (common) identity of both $\mathfrak{A}$ and $\mathfrak{B}$. We use M^- to denote $\{ x \in M : x \neq e \}$, the non-identity elements of a polygroup $\mathfrak{M}$. A new system $\mathfrak{A}[\mathfrak{B}] = \langle M, *, e, ^I \rangle$, called the *extension of* $\mathfrak{A}$ *by* $\mathfrak{B}$, is formed in the

following way. Set $M = A^- \cup B^- \cup \{e\}$ and let $e^I = e$, $x^I = x^{-1}$ (in the appropiate system), $e*x = x*e = x$ for all $x \in M$, and for all $x,y \in M^-$,

$$x*y = \begin{cases} x\cdot y & \text{if } x,y \in A \\ x & \text{if } x \in B,\ y \in A \\ y & \text{if } x \in A,\ y \in B \\ x\cdot y & \text{if } x,y \in B,\ y \neq x^{-1} \\ x\cdot y \cup A & \text{if } x,y \in B \text{ and } y = x^{-1}. \end{cases}$$

In the last clause, e occurs in both $x\cdot y$ and A. If $A = \{e,a_1,a_2,\ldots\}$ and $B = \{e,b_1,b_2,\ldots\}$, the table for $*$ in $\mathfrak{A}[\mathfrak{B}]$ has the form

	e	a_1	a_2	...	b_1	b_2	...
e	e	a_1	a_2	...	b_1	b_2	...
a_1	a_1	a_1a_1	a_1a_2	...	b_1	b_2	...
a_2	a_2	a_2a_1	a_2a_2	...	b_1	b_2	...
.	.	.	.	...	.	.	...
.	.	.	.	...	.	.	...
.	.	.	.	...	.	.	...
b_1	b_1	b_1	b_1	...	b_1*b_1	b_1*b_2	...
b_2	b_2	b_2	b_2	...	b_2*b_1	b_2*b_2	...
.	.	.	.	...	.	.	...
.	.	.	.	...	.	.	...
.	.	.	.	...	.	.	...

Several special cases of the algebra $\mathfrak{A}[\mathfrak{B}]$ are useful. Before describing them we need to assign names to the two 2-element polygroups. Let 2 denote the group Z_2 and let 3 denote the polygroup $S_3 /\!/ \langle(12)\rangle \cong Z_3 /\!/ \theta$ where θ is the special conjugation with blocks $\{0\}\{1,2\}$. The multiplication table for 3 is

	0	1
0	0	1
1	1	0,1

The names 2 and 3 are suggested by the color schemes that represent the algebras (see Section 3).

EXAMPLE 1. Adjoining a new identity element.

The system $3[\mathfrak{M}]$ is the result of adding a "new" identity element to the polygroup $\mathfrak{M}$. The system $2[\mathfrak{M}]$ is almost as good. For example, suppose $\mathfrak{R}$ is the system with table

	0	1	2
0	0	1	2
1	1	02	12
2	2	12	01

Then

	0	a	1	2
0	0	a	1	2
a	a	0a	1	2
1	1	1	0a2	12
2	2	2	12	0a1

3[𝔑]

	0	a	1	2
0	0	a	1	2
a	a	0	1	2
1	1	1	0a2	12
2	2	2	12	0a1

2[𝔑]

The element "a" acts like the "old" identity on 𝔑.

EXAMPLE 2. Adding a "last" element.

In section 20 of [1] two non-isomorphic one-element extensions of a polygroup 𝔐 were introduced. In the present terminology these algebras are just 𝔐[2] and 𝔐[3]. For example, the tables for 𝔑[2] and 𝔑[3] are given below.

	0	1	2	a
0	0	1	2	a
1	1	02	12	a
2	2	12	01	a
a	a	a	a	012

𝔑[2]

	0	1	2	a
0	0	1	2	a
1	1	02	12	a
2	2	12	01	a
a	a	a	a	012a

𝔑[3]

EXAMPLE 3. As an example of 𝔄[𝔅] where neither 𝔄 nor 𝔅 are minimal systems we consider 𝔑[𝔑] whose table is given below.

	0	1	2	a	b
0	0	1	2	a	b
1	1	02	12	a	b
2	2	12	01	a	b
a	a	a	a	012b	ab
b	b	b	b	ab	012a

We finish this section by showing that the extension construction will always yield a polygroup.

THEOREM 1. $\mathfrak{A}[\mathfrak{B}]$ *is a polygroup.*

Proof. Since (P_2) is clear it is enough to check (P_1) and (P_3).

(P_1) : $(x*y)*z = x*(y*z)$.

Without loss of generality we may assume $x,y,z \neq e$ and not all elements belong to A. Note that

(1) if $u \in B$ and $v \in A$, then $u*v = v*u = u$.

If exactly one of x,y,z belong to B, then (1) implies that both sides of (P_1) equal the element in $\{x,y,z\} \cap B$. If exactly two of x,y,z belong to B, say u and v, then (1) implies that both sides of (P_1) equal $u*v$. We assume $x,y,z \in B^-$ and show

(2) $u \in (x*y)*z$ implies $u \in x*(y*z)$.

If $u \notin A$, then $u \in w*z$ for some $w \in x*y$. Now, if $w \notin A$, $w \in x \cdot y$ and $u \in w \cdot z$ so

$$u \in (xy)z = x(yz) \quad (\text{in } B) \subseteq x*(y*z).$$

Also, if $w \in A$, $u \in w*z = z$ (so $u = z$) and $e \in xy$. Thus,

$$u = z \in (xy)z = x(yz) \subseteq x*(y*z).$$

Now, suppose $u \in A$. Then $z^{-1} \in x*y$. $z^{-1} \notin A$ so $z^{-1} \in xy$ (in B), so $e \in (xy)z = x(yz)$. Thus,

$$x^{-1} \in y \cdot z \subseteq y*z \text{ and hence } u \in A \subseteq x*(y*z).$$

The proof of the opposite inclusion $x*(y*z) \subseteq (x*y)*z$ is similar to (2).

(P_3) : $x \in y*z$ implies $y \in x*z^I$ and $z \in y^I*x$.

The condition is clear if $x,y,z \in A$. Since $x \in B^-$ implies y or z belongs to B^- and $x \in A$ implies $z \in B^-$, we may assume at least two of x,y,z belong to B^-. On the other hand, if $x,y,z \in B^-$, then $x \in y*z$ implies $x \in y \cdot z$ (in B) from which (P_3) follows. Therefore we may assume exactly two of x,y,z belong to B^-. This reduces to two cases.

(3) $x \in y*z$ where $x,y \in B^-$ and $z \in A$.

By (1), $y*z = y$ so $x = y$; thus $y = x = x*z^{-1}$ using (1) again and $z \in A \subseteq x^{-1}*x = y^{-1}*x$.

(4) $x \in y*z$ where $x \in A$ and $y,z \in B^-$.

In this case $y = z^{-1}$ so the desired conclusion follows using (1). This completes the proof of (P_3) and hence the theorem.

Where there is no confusion possible we write $* = \cdot$ and ${}^{I} = {}^{-1}$.

3. PROPERTIES OF $\mathfrak{A}[\mathfrak{B}]$. The first result shows that the extension of $\mathfrak{A}$ by $\mathfrak{B}$ preserves being chromatic.

THEOREM 2. *If* $\mathfrak{A} \cong \mathfrak{M}_{\mathcal{V}}$ *and* $\mathfrak{B} \cong \mathfrak{M}_{\mathcal{W}}$, *then* $\mathfrak{A}[\mathfrak{B}]$ *is also chromatic.*

Proof. Suppose $\mathcal{V} = \langle V, C_a \rangle_{a \in A^-}$. First introduce a family of pairwise disjoint color schemes $\{ \mathcal{V}_w : w \in W \}$ where each $\mathcal{V}_w$ is isomorphic to $\mathcal{V}$. Assume the vertex set of $\mathcal{V}_w$ is V_w and the isomorphism of $\mathcal{V}$ onto $\mathcal{V}_w$ sends x to x_w. We construct a scheme $\mathcal{V}[\mathcal{W}]$ in the following way. Replace each vertex w of the scheme $\mathcal{W}$ by the copy of $\mathcal{V}$ with vertex set V_w. Thus the set of all vertices of $\mathcal{V}[\mathcal{W}]$ is just the union of all V_w's. An edge coloring using the elements of $A^- \cup B^-$ as colors is introduced in the following way. For $a \in A^-$ and $b \in B^-$ let

$$(x_u, y_v) \in C_a \quad \text{iff} \quad u = v \quad \text{and} \quad (x,y) \in C_a \quad (\text{in } \mathcal{V}_u),$$

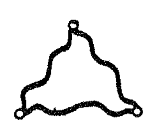
$$(x_u, x_v) \in C_b \quad \text{iff} \quad (u,v) \in C_b \quad (\text{in } \mathcal{W}).$$

It is easily seen that $\mathcal{V}[\mathcal{W}]$ is a scheme that represents $\mathfrak{A}[\mathfrak{B}]$.

The converse of Theorem 2 will be established later (Theorem 6).

The construction given in the proof above can be carried out in practice. The idea is to take a color scheme representing $\mathfrak{B}$ and "blow-it-up" by replacing each vertex by the configuration that represents $\mathfrak{A}$. As an illustration we use this method to produce representations for the systems $2[\mathfrak{R}]$, $3[\mathfrak{R}]$, $\mathfrak{R}[2]$, $\mathfrak{R}[3]$, and $\mathfrak{R}[\mathfrak{R}]$ given in Examples 1,2,3. The systems 2, 3, and $\mathfrak{R}$ have representations given as follows.

2



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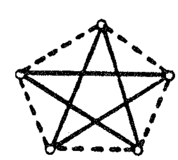
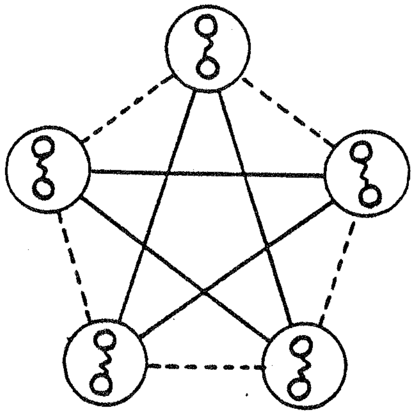
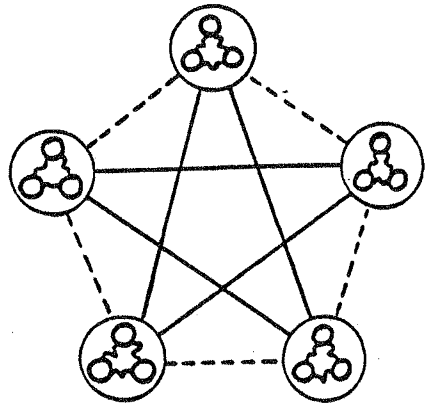


$\mathfrak{R}$

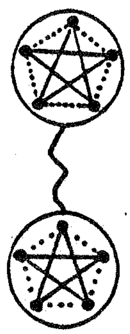
The method of Theorem 2 yields:



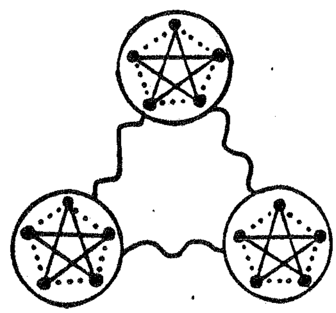
2[$\mathfrak{R}$]



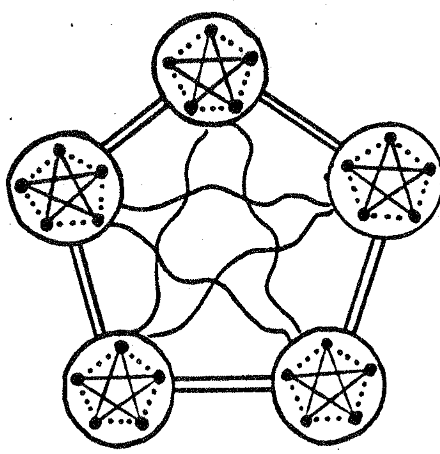
3[$\mathfrak{R}$]



$\mathfrak{R}$[2]



$\mathfrak{R}$[3]



$\mathfrak{R}$[$\mathfrak{R}$]

The next three results show that various important special classes of chromatic polygroups are closed under the extension operation $\mathfrak{A}[\mathfrak{B}]$. Several types of proofs are offered. The approach via automorphism groups, illustrated by the proof of Theorem 3, could be used to establish all three theorems. A more concrete approach is used to prove Theorems 4 and 5. When using a special representation of a polygroup it is often convenient to explicitly give the groups and/or conjugations involved.

THEOREM 3. *If* $\mathfrak{A}, \mathfrak{B} \in Q^2(\text{Group})$, *then* $\mathfrak{A}[\mathfrak{B}] \in Q^2(\text{Group})$.

Proof. Recall (Theorem 4.1 of [2]) that for a polygroup $\mathfrak{M}$,

(*) $\mathfrak{M} \in Q^2(\text{Group})$ iff $\mathfrak{M} \cong \mathfrak{M}_{\mathcal{V}}$ where $\text{Aut}(\mathcal{V})$ is transitive on vertices.

Suppose $\mathfrak{A} \cong \mathfrak{M}_{\mathcal{V}}$, $\mathfrak{B} \cong \mathfrak{M}_{\mathcal{W}}$, and $\mathcal{V}[\mathcal{W}]$ is the color scheme constructed from $\mathcal{V}$ and $\mathcal{W}$ in the proof of Theorem 2. Automorphisms τ on $\mathcal{V}$ and σ on $\mathcal{W}$ induce automorphisms on $\mathcal{V}[\mathcal{W}]$ in the following way.

(i) For $\sigma \in \text{Aut}(\mathcal{W})$ define $\hat{\sigma}$ on $\mathcal{V}[\mathcal{W}]$ by $\hat{\sigma}(x_w) = x_{\sigma(w)}$ for all $w \in W$.

(ii) For $\tau \in \text{Aut}(\mathcal{V})$ and $w \in W$, define $\bar{\tau}_w$ on $\mathcal{V}[\mathcal{W}]$ so that $\bar{\tau}_w$ acts like τ on $\mathcal{V}_w$ and is the identity otherwise.

It is easily seen that the maps $\hat{\sigma}$ and $\bar{\tau}_w$ described in (i) and (ii) are automorphisms of $\mathcal{V}[\mathcal{W}]$. From (*) we may assume $\text{Aut}(\mathcal{V})$ and $\text{Aut}(\mathcal{W})$ are transitive. Using maps of type (i) and (ii) it easily follows that $\text{Aut}(\mathcal{V}[\mathcal{W}])$ is transitive on vertices so (*) yields the desired conclusion.

THEOREM 4. *If* $\mathfrak{A}, \mathfrak{B} \in Q^2_s(\text{Group})$, *then* $\mathfrak{A}[\mathfrak{B}] \in Q^2_s(\text{Group})$.

Proof. Suppose $\mathfrak{A} = G_1/\theta_1$ and $\mathfrak{B} = G_2/\theta_2$ where θ_1 and θ_2 are special conjugations on G_1 and G_2 respectively. Let $G = G_1 \times G_2$ and define θ on G by

$$(g_1,g_2)\theta(g_1',g_2') \text{ iff } (g_2=g_2'=e \text{ and } g_1\theta_1 g_1') \text{ or } (g_2,g_2' \neq e \text{ and } g_2\theta_2 g_2').$$

Note that the θ-classes are

$$\theta(g,e) = \{ (h,e) : h\theta_1 g \}$$

and, for $h \neq e$,

$$\theta(e,h) = \{ (g,h') : h'\theta_2 h \}.$$

To show θ is a special conjugation, conditions (i), (ii), and (iii) need to be checked. First, (iii) holds because θ_1 special implies $\theta(e,e) = \{(e,e)\}$. Also (i), $\theta(g,h)^{-1} = (\theta(g,h))^{-1}$, holds since θ_1 and θ_2 have similar properties. It remains to check

(ii) $\theta((g_1,h_1)(g_2,h_2)) \subseteq (\theta(g_1,h_1))(\theta(g_2,h_2))$.

Suppose $(g,h) \in \theta(g_1g_2,h_1h_2) = \theta((g_1,h_1)(g_2,h_2))$. The definition of θ gives two cases:

Case 1. $h = h_1h_2 = e$ and $g\theta_1(g_1g_2)$.

Since θ_1 is a conjugation, $g = g_1'g_2'$ for some $g_1'\theta_1 g_1$ and $g_2'\theta_1 g_2$. Then $(g,e) = (g_1',h_1)(g_2',h_2)$ so it suffices to show that $(g_i',h_i)\theta(g_i,h_i)$ for $i=1,2$. The conclusion follows from $g_i'\theta_1 g_i$ when $h_1=h_2=e$ while it follows from $h_i\theta_2 h_i$ if

h_1 , $h_2 \neq e$.

Case 2. h, $h_1h_2 \neq e$ and $h \,\theta_2\, h_1h_2$.

Since θ_2 is a conjugation, $h = h_1'h_2'$ for some $h_1' \theta_2 h_1$ and $h_2' \theta_2 h_2$. This yields $(g,h_1')\,\theta\,(g_1,h_1)$ and $(e,h_2')\,\theta\,(g_2,h_2)$ whenever $h_1',h_2' \neq e$; so

$$(g,h) = (g,h_1')(e,h_2') \in (\theta(g_1,h_1))(\theta(g_2,h_2)).$$

On the other hand, suppose one of h_1', h_2' is e, say $h_1' = e$ and $h_2' = h \neq e$. Then $(g_1,e)\,\theta\,(g_1,h_1)$ and $(g_1^{-1}g,h)\,\theta\,(g_2,h)$ since $h \neq e$ from which it follows that (g,h) belongs to $(\theta(g_1,h_1))(\theta(g_2,h_2))$.

Thus, θ is a special conjugation on the group $G_1 \times G_2$. A bijection F between the elements of $\mathfrak{A}[\mathfrak{B}]$ and $G_1 \times G_2/\theta$ is defined in the following way. Let $F(e) = \theta(e,e)$ and, for $a = \theta_1 g_1 \neq e$ in A^- let

$$F(a) = \theta(g_1,e),$$

and, for $b = \theta_2 g_2 \neq e$ in B^-, let

$$F(b) = \theta(e,g_2).$$

From the description of the θ-classes above it is clear that F maps $\mathfrak{A}[\mathfrak{B}]$ one-one onto $G_1 \times G_2/\theta$. By properties (i) and (ii) of θ the inverses and identity elements correspond. Computations, as in the proof of (ii), show that F preserves products in case at least one factor belongs to $\mathfrak{A}$. When both factors belong to $\mathfrak{B}$ there are two cases. First, if $\theta_2 g_2, \theta_2 g_2' \in B^-$ and $\theta_2 g_2 \neq (\theta_2 g_2')^{-1}$, then

$$\begin{aligned}
F(\theta_2 g_2, \theta_2 g_2') &= F(\{\theta_2 g : e \neq g \in (\theta_2 g_2)(\theta_2 g_2')\}) \\
&= \{\theta(e,g) : g \in (\theta_2 g_2)(\theta_2 g_2')\} \\
&= \{(h,g) : g \in (\theta_2 g_2)(\theta_2 g_2')\} \\
&= (\theta(e,g_2))(\theta(e,g_2')) \\
&= F(\theta_2 g_2)F(\theta_2 g_2')
\end{aligned}$$

Finally, suppose $\theta_2 g_2, \theta_2 g_2' \in B^-$ and $\theta_2 g_2' = (\theta_2 g_2)^{-1}$. Then $e \in (\theta_2 g_2)(\theta_2 g_2')$ so

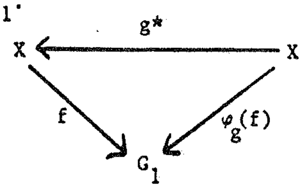
$$\begin{aligned}
F((\theta_2 g_2)(\theta_2 g_2')) &= F(\{\theta_2 g : g \in (\theta_2 g_2)(\theta_2 g_2')\} \cup A \\
&= \{\theta(e,g) : g \in (\theta_2 g_2)(\theta_2 g_2')\} \cup (G_1 \times \{e\}) \\
&= \{(h,g) : g \in (\theta_2 g_2)(\theta_2 g_2')\} \quad \text{since } e \in (\theta_2 g_2)(\theta_2 g_2') \\
&= (\theta(e,g_2))(\theta(e,g_2')) \\
&= F(\theta_2 g_2)\,F(\theta_2 g_2').
\end{aligned}$$

The theorem follows from the fact F is an isomorphism.

The next result shows the class of double coset algebras is closed under the extension construction. For information on semi-direct products see M.Hall [4]. The group $\hat{G}$ defined below is also known as the wreath product of G_1 and G_2, see e.g., H. Neumann [7], p.45ff.

THEOREM 5. *If* $\mathfrak{A} \cong G_1/H_1$ *and* $\mathfrak{B} \cong G_2/H_2$, *then there exist groups* $\hat{G}$ *and* $\hat{H}$ *with* $\hat{G} \supseteq \hat{H}$ *such that* $\mathfrak{A}[\mathfrak{B}] \cong \hat{G}/\hat{H}$.

<u>Proof</u>. Let $X = G_2/H_2 = \{H_2 g : g \in G_2\}$ and let $\hat{G} = G_1^X \circledS_\varphi G_2$, the semi-direct product of G_1^X by G_2, where φ, mapping G_2 into $\mathrm{Aut}(G_1^X)$, is the homomorphism given by $\varphi_g(f) = g^*f$ for all $f \in G_1^X$.



I.e., $g \in G_2$ induces $g^*: X \longrightarrow X$ by right multiplication, so $\varphi_g(f)(x) = f(xg)$ for $g \in G_2$, $f \in G_1^X$ and $x \in X$. Also, let $\hat{H} = (H_1 \times G_1^{X-\{H_2\}}) \circledS_\varphi H_2$ where

$$\varphi : H_2 \longrightarrow \mathrm{Aut}(\, H_1 \times G_1^{X-\{H_2\}})$$

is defined, as above, by $\varphi_g(f) = g^*f$.

Note that $\hat{H} = \bar{H} \circledS_\varphi H_2$ where $\bar{H} = \{f \in G_1^X : f(H_2) \in H_1\}$. Clearly $\hat{H}$ is a subgroup of $\hat{G}$, so it remains to show that $\mathfrak{A}[\mathfrak{B}] \cong \hat{G}/\hat{H}$.

First we identify $\mathfrak{A}$ with part of $\hat{G}/\hat{H}$. For $g \in G_1$ let $\hat{g} = (f,1)$ where 1 is the identity element of G_2 and $f \in G_1^X$ is defined by

$$f(H_2 x) = \begin{cases} g & \text{if } H_2 x = H_2 \\ e & \text{if } H_2 x \neq H_2 \end{cases}$$

Then,

$$\begin{aligned} \hat{H}\hat{g}\hat{H} &= (\bar{H} \circledS H_2)(f,1)(\,\bar{H} \circledS H_2) \\ &= (\bar{H}f \circledS H_2)(\bar{H} \circledS H_2) \\ &= (\bar{H}\, f\, \bar{H}) \circledS H_2 \end{aligned}$$

where the second equality holds because, for $h \in H_2$, φ_h fixes the "H_2-coordinate" of f. Thus,

(1) $(G_1^X \circledS H_2)/\hat{H} \cong G_1/H_1$.

Now we consider the elements in G_2/H_2 ($\cong \mathfrak{B}$). For $g \in G_2$ let $\bar{g} = (E,g)$ where E is the identity element of G_1^X. Then

$$\hat{H}\bar{g} = (\,\bar{H} \circledS H_2)\bar{g} = \bar{H} \circledS H_2 g$$

and, for $g \notin H_2$,

$$\hat{H}\bar{g}\hat{H} = (\bar{H} \circledS H_2 g)(\bar{H} \circledS H_2) = G_1^X \circledS (H_2 g H_2)$$

since, for $f_1, f_2 \in \bar{H}$, $(f_1 \cdot f_2^{hg})(x) = f_1(x) f_2(xhg)$ will produce any element of G_1^X. (To see this observe that $g \notin H_2$ means g^* is a permutation of X that moves H_2; so, for all $x \in X$, either $f_1(x)$ or $f_2(xhg)$ can be any element of G_1.) Thus,

(2) $\hat{G} = \bigcup \{G_1^X \circledS b \; : \; b \in G_2/H_2\}$.

A one-one correspondence between the non-identity elements of $(G_1/H_1)[G_2/H_2]$ and $\hat{G}/\hat{H}$ is introduced as follows: to $a \in (G_1/H_1)^-$ assign the element $\bar{a} \circledcirc H_2$ where $\bar{a} = \{ f \in G_1^X : f(H_2) \in a \}$ and to $b \in (G_2/H_2)^-$ correlate $G_1^X \circledcirc b$. In view of (1), to show this correspondence is an isomorphism it is enough to check:

(3) $(\hat{H}\hat{g}_1\hat{H})(\hat{H}\bar{g}_2\hat{H}) = \hat{H}\bar{g}_2\hat{H}$ for $g_1 \in G_1$, $g_2 \in G_2 - H_2$,

(4) $(\hat{H}\bar{g}_2\hat{H})(\hat{H}\hat{g}_1\hat{H}) = \hat{H}\bar{g}_2\hat{H}$ for $g_1 \in G_1$, $g_2 \in G_2 - H_2$, and

(5) $(\hat{H}\bar{g}_1\hat{H})(\hat{H}\bar{g}_2\hat{H}) = G_1^X \circledcirc (H_2 g_1 H_2 g_2 H_2)$ for $g_1, g_2 \in G_2 - H_2$.

To establish (3), let $\hat{g}_1 = (g_1', 1)$. Then

$$(\hat{H}\hat{g}_1\hat{H}) \cdot (\hat{H}\bar{g}_2\hat{H}) = ((\bar{H}g_1'\bar{H}) \circledcirc H_2)(G_1^X \circledcirc H_2 g_2 H_2)$$
$$= G_1^X \circledcirc H_2 g_2 H_2$$
$$= \hat{H}\bar{g}_2\hat{H}$$

since, for $h \in H_2$, φ_h fixes the "H_2-coordinate" and permutes all others.

The verification of (4) is easier:

$$(\hat{H}\bar{g}_2\hat{H})(\hat{H}\hat{g}_1\hat{H}) = (G_1^X \circledcirc H_2 g_2 H_2)((\bar{H}g_1'\bar{H}) \circledcirc H_2) = G_1^X \circledcirc H_2 g_2 H_2.$$

Finally we check (5):

$$(\hat{H}\bar{g}_1\hat{H})(\hat{H}\bar{g}_2\hat{H}) = (G_1^X \circledcirc H_2 g_1 H_2)(G_1^X \circledcirc H_2 g_2 H_2) = G_1^X \circledcirc (H_2 g_1 H_2)(H_2 g_2 H_2).$$

It now follows that $G/H \cong \mathfrak{A}[\mathfrak{B}]$ as desired.

We now consider the converses of the properties established in Theorems 2,3,4, and 5. The basic idea for establishing the converses is illustrated by the proof of the following result.

THEOREM 6. *If* $\mathfrak{A}[\mathfrak{B}]$ *is chromatic, then both* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are chromatic.*

Proof. Suppose $\mathfrak{A}[\mathfrak{B}] \cong \mathfrak{M}_{\mathscr{W}}$ for some color scheme $\mathscr{W} = \langle W, C_x \rangle_{x \in C}$. Recall that $C = A^- \cup B^-$. Define a relation $\approx$ on W by

$$w \approx w' \text{ iff } w = w' \text{ or } (w,w') \in C_a \text{ for some } a \in A^-.$$

It is easily seen that $\approx$ is an equivalence relation on W and each $\approx$-block, say $[p] = \{w : w \approx p\}$ for a fixed $p \in W$, inherits the structure of a color scheme from $\mathscr{W}$. The color algebra of this scheme is exactly $\mathfrak{A}$; so $\mathfrak{A}$ is chromatic.

In order to treat $\mathfrak{B}$ we form a new scheme $\mathscr{W}/\approx$ on the set $\{[w] : w \in W\}$ using the elements of B^- as colors. For distinct vertices [v] and [w] set

$$([v],[w]) \in C_b \quad \text{iff} \quad (v,w) \in C_b \quad (\text{in } \mathscr{W}).$$

Since $a_1 b a_2 = b$ holds in $\mathfrak{A}[\mathfrak{B}]$ for $a_1, a_2 \in A$ and $b \in B$, it follows that the assignment of a color to the edge $([v],[w])$ is independent of the $\approx$-representation. It is not hard to check that $\mathscr{W}/\approx$ is a color scheme and $\mathfrak{M}_{\mathscr{W}/\approx} \cong \mathfrak{B}$ as desired.

Using the idea above an analysis of the proofs of Theorems 3,4, and 5 give a hint of how to construct their converses. We leave the details to the reader.

THEOREM 7. *If* $\mathfrak{A}[\mathfrak{B}]$ *is a double coset algebra (in* Q^2(Group), Q_s^2(Group) *), then both* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are double coset algebras (in* Q^2(Group), Q_s^2(Group) *respectively).*

4. AN APPLICATION. We conclude with an easy application of the extension construction to the study of relation algebras. There are many non-chromatic polygroups with 4 elements - at least 28 and at most 34. As one example we cite the algebra $\mathfrak{N}_0$ with multiplication table:

	0	1	2	3
0	0	1	2	3
1	1	1	0123	13
2	2	0123	2	23
3	3	13	23	012

In view of the connection between polygroups and integral relation algebras (see [2]) the fact that $\mathfrak{N}_0$ is non-chromatic is just the result of McKenzie [6] that the corresponding relation algebra is non-representable. $\mathfrak{N}_0$ can also be shown to be non-chromatic by a direct argument.

In Section 2 four extensions, $\mathfrak{M}[2]$, $\mathfrak{M}[3]$, $2[\mathfrak{M}]$, and $3[\mathfrak{M}]$ were given that add a new element to a polygroup $\mathfrak{M}$. By Theorems 2 and 6, $\mathfrak{M}$ is chromatic if and only if each extension is chromatic. Starting with $\mathfrak{N}_0$, McKenzie's example above, we can obtain a sequence (in fact many sequences) of non-chromatic polygroups. For example,

$$\mathfrak{N}_1 = \mathfrak{N}_0[2], \quad \mathfrak{N}_2 = \mathfrak{N}_1[2], \ \ldots$$

Again using the connection [2] between polygroups and relation algebras we obtain:

PROPOSITION 8. *For all* $n \geq 4$ *there exist a non-representable integral relation algebra with* n *atoms.*

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