Constructions of Color Schemes

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Representation techniques for an important class of multigroups, called polygroups, are discussed. For a set C of colors and an involution ι of C, a color scheme is a system $\langle V, \{C_a : a \in C\}\rangle$ such that (i). $\{C_a : a \in C\}$ is a partition of $\{(x, y) \in V^2 : x \neq y\}$, (ii). $C_a^{\cup} = C_{\iota(a)}$ for each $a \in C$, (iii). each color is present on some edge emininating from each vertex, and (iv). for $a, b, c \in C$, $C_c \cap (C_a \mid C_b) \neq \emptyset$ implies $C_c \subseteq C_a \mid C_b$. A multigroup, called a chromatic polygroup, is associated with a color scheme in a natural way. Three techniques for showing that a polygroup is chromatic are discussed: (1) the construction of random schemes with forbidden configurations, (2) the construction of iterative 1-point extensions, and (3) product constructions.

Jsou diskutovány techniky reprezentace důležité třídy multigrup, zvané polygrupy. Chromatická polygrupa je multigrupa, asociovaná přirozeným způsobem s barevným schématem. Jsou diskutovány tři techniky pro demonstraci chromatičnosti polygrupy: (1) konstrukce náhodných schémat se zakázanými konfiguracemi, (2) konstrukce iterativních jednobodových extenzí, a (3) produktové konstrukce.

Исследуются техники представления одного важного класса мултигрупп, называемых полыгруппы. Хроматическая полыгруппа — ето мултигруппа, ассоцированная натуральным способом с цветной схемой. Нсследуются три техники установления хроматичности данной полыгруппы: (1) построения случайных схем с запрещенными конфугурациями, (2) построения итеративных одноточечных расширений, и (3) построения при помощи произведений.

A polygrup is a multivalued group-like system $\mathfrak{M} = \langle \mathfrak{M}, \cdot, ^{-1}, e\rangle$ where $e \in \mathfrak{M}$, $^{-1}$ is a unary operation on $\mathfrak{M}$, $\cdot$ maps $\mathfrak{M}^2$ into nonempty subsets of $\mathfrak{M}$, and the following axioms hold for all $x, y, z \in \mathfrak{M}$:

(i) $(x . y) . z = x . (y . z)$,

(ii) $e . x = x = x . e$,

(iii) $x \in y . z$ implies $y \in x . z^{-1}$ and $z \in y^{-1} . x$.

Natural examples of polygroups can be derived from systems such as geometries, groups, distance transitive graphs, and association schemes (see [1]). Moreover,

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polygroups are closely connected to the theory of relations (see [2]) and to other forms of algebraic logic.

This paper is concerned with a class of polygroups related to a strong coloring of complete graphs. Given a set C (of colors) and an involution ι of C, a *color scheme* is a system $\mathscr{V} = \langle V, \{c_a : a \in C\}\rangle$ such that

(i) $\{C_a : a \in C\}$ partitions $V^2 - I = \{(x, y) \in V^2 : x \neq y\}$,

(ii) $C_a^{\cup} = C_{\iota(a)}$ for $a \in C$,

(iii) each color is present on some edge eminating from each vertex,

(iv) for $a, b, c \in C$, $C_c \cap (C_a \mid C_b) \neq \emptyset$ implies $C_c \subseteq C_a \mid C_b$.

The algebra of a color scheme $\mathscr{V}$ is a system $\mathfrak{M}_{\mathscr{V}} = \langle C \cup \{I\}, *, ^{-1}, I\rangle$ where $I \neq C$, $a^{-1} = \iota(a)$, $I^{-1} = I$, $I \,.\, a = a \,.\, I$, and for $a, b \in C$,

$$a * b = \{c \in C : C_c \cap (C_a \mid C_b) \neq \emptyset\} \cup \{I : a = b^{-1}\} .$$

A polygroup isomorphic to some $\mathfrak{M}_{\mathscr{V}}$ is called *chromatic*. Chromatic polygroups are closely related to representable relation algebras (see [1]).

A basic problem for a given polygroup $\mathfrak{M}$ is to determine whether or not it is chromatic. If it is, it becomes of interest to know whether $\mathfrak{M}$ has certain additional properties.

(1) Is $\mathfrak{M} \cong G /\!/ H$, the polygroup of all double cosets of a subgroup H of a group G?

(2) Is $\mathfrak{M} \cong \mathfrak{M}_{\mathscr{V}}$ for a homogeneous coherent configuration?

(3) Is $\mathfrak{M} \in Q^2(\text{Group})$, i.e., isomorphic to $G /\!/ \Theta$ where Θ is a conjugation on some group G?

The purpose of this article is to discuss in one place several recent methods that can be used to show that a polygroup is chromatic or has one of the other properties above. In Section 1 a general iterative procedure is used to build color schemes with a prescribed set of forbidden configurations. A strong form of this inductive process leads to a type of random scheme in section 2 which contains an easy sufficient condition for the existence of these schemes. The consideration of special types of color scheme representations leads in a natural way to closure properties for the associated classes of polygroups. In section 3 closure properties with respect to two types of product operations are studied. A reader should consult [1] for terminology and notation that is not obvious.

1. Iterative 1-point Extensions

The idea behind this method is the observation that a polygroup $\mathfrak{M}_{\mathscr{V}}$ specifies which configurations of $\mathscr{V}$ are forbidden and which are admissible.

An element of C^3 is called a *colored triangle* (or just a *triangle*). A triangle (i, j, k) is *realizable in* $\mathscr{V}$ *on an edge* $(x, y) \in V^2 - I$ if $(x, y) \in C_k \cap (C_i \mid C_j)$.

A class K of colored triangles is *locally realizable* in $\mathscr{V}$ if, for every edge $(x, y) \in C_k$ and every $(i, j, k) \in K$, $(x, y) \in C_i \mid C_j$. A class K of triangles is *forbidden* in $\mathscr{V}$ if no triangle in K is realizable on an edge in $\mathscr{V}$.

The forbidden class $FC(\mathfrak{M})$ of a polygroup $\mathfrak{M}$ is the class

$$FC(\mathfrak{M}) = \{(a, b, c) \in (M \setminus \{e\})^3 : c \notin a \,.\, b\}$$

of colored triangles. When explicitly writing $FC(\mathfrak{M})$ we omit the obvious symmetrics. For example, we say, the algebra A_1^{0123} in figure 1 has forbidden class $\{(1, 2, 3)\}$ instead of $\{(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)\}$.

	0	1	2	3
0	0	1	2	3
1	1	0123	12	13
2	2	12	0123	23
3	3	13	23	0123

Figure 1.

A system $\mathscr{V}$ that satisfies color scheme axioms (i) and (ii) is called a *partial scheme*. The following lemma provides the key to constructing color schemes having a prescribed forbidden class.

Lemma 1. $\mathfrak{M}$ is chromatic iff there a partial scheme that forbids $FC(\mathfrak{M})$ and locally realizes it complement.

Proof. Suppose $\mathscr{V}$ is a partial scheme with the property of the lemma. Assume there is some edge in C_j into a vertex $x \in V$. For each $i \in M$, $j \,.\, i \neq \emptyset$ so some triangle (j, i, k) will be locally realizable in $\mathscr{V}$. Using properties of $\mathfrak{M}$ an edge $(x, z) \in C_j$ can be found which establishes condition (iii) of the definition of color scheme. Condition (iv) is immediate from the assumption on $\mathscr{V}$.

A color scheme produced by the method described in the next theorem is called an iterative 1-point extension.

Theorem 2. A finite polygroup $\mathfrak{M}$ is chromatic if $K = FC(\mathfrak{M})$ has the property

$$(*)\begin{cases}\text{for every finite partial scheme } \mathscr{V} = \langle V, C_a \rangle_{a \in C} \text{ that forbids } K\text{, every edge}\\ (x, y) \in C_k \text{ and every triangle } (i, j, k) \notin K\text{, there exist a finite partial scheme}\\ \mathscr{V}' \text{ extending } \mathscr{V} \text{ such that } \mathscr{V}' \text{ forbids } K \text{ and } (i, j, k) \text{ is realized on } (x, y) \in \mathscr{V}'.\end{cases}$$

Proof. A scheme that represents $\mathfrak{M}$ is constructed in a countable number of steps. Any finite partial scheme that forbids K is taken as the first stage. The partial scheme $\mathscr{V}_{n+1}$ for the $n + 1$ stage is obtained from the scheme $\mathscr{V}_n$ at the nth stage by using $(*)$ repeatedly (but only finitely many many times) to realize all admissible triangles on edges in $\mathscr{V}_n$. The union of all $\mathscr{V}_n$'s has the property of Lemma 1. Complete details may be found in [3].

The procedure above is the most general sufficient condition known. However, it has several disadvantages.

(1) It is often a tedious, arduous task to verify $(*)$.

(2) Through the inductive procedure one loses control of the properties of the color scheme that ultimately represents $\mathfrak{M}$. To some extent this cannot be improved. For example in [3] Theorem 2 was used to show that a system $\mathfrak{N}_1$ was chromatic although $\mathfrak{N}_1$ is not isomorphic to the algebra of a finite homogeneous coherent configuration; in particular $\mathfrak{N}_1$ is not a double coset algebra of a finite group.

Condition $(*)$ in Theorem 2 is not necessary for $\mathfrak{M}$ to be chromatic. This is a consequence of the next two results about the algebra A_{14}^{01} with forbidden class $\{(1, 1, 2), (1, 1, 3), (2, 2, 2), (3, 3, 3)\}$ (see figure 2).

	0	1	2	3
0	0	1	2	3
1	1	01	23	23
2	2	23	013	123
3	3	23	123	012

A_{14}^{01}

Figure 2

The construction below was worked out jointly with Bill Sands.

Proposition 3. A_{14}^{01} is chromatic.

Proof. Partition R^+ (= the positive reals) into three disjoint subsets B_0, B_1, B_2 such that each B_i is dense in R^+, cofinal in R^+, and coinitial in R^+. Introduce $\mathscr{V} = \langle R^+, C_1, C_2, C_3 \rangle$ where, for $x, y \in R^+$, $x \neq y$

$(x, y) \in C_1$ iff $x, y \in B_i$ for some i.

$(x, y) \in C_2$ iff for some i, $m \in B_i$ and $M \in B_j$ where $j \equiv i + 1 \pmod 3$, $m = \min\{x, y\}$ and $M = \max\{x, y\}$.

$(x, y) \in C_3$ iff $(x, y) \notin C_1$ and $(x, y) \notin C_2$.

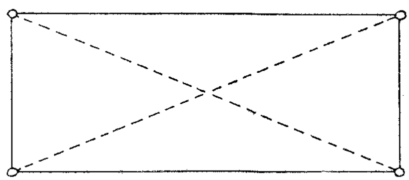
Using the properties of the B_i's it is not difficult to check that $\mathscr{V}$ is a color scheme with the desired forbidden class.

Proposition 4. If $\mathscr{V}$ is a color scheme with $\mathfrak{M}_{\mathscr{V}} \cong A_{14}^{01}$, then $\mathscr{V}$ is 3-partite in C_1, (i.e., there are 3 C_1-cliques).

Proof. Since $1.1 = \{0, 1\}$, $\mathscr{V}$ is n-partite for some n and since (1, 2, 3)-triangles are realizable $n \geqq 3$. Fix $(x, y) \in C_1$, and choose z with $(x, z) \in C_2$, $(y, z) \in C_3$. [This realizes a (2, 3, 1)-triangle on (x, y).]. Choose u with $(y, u) \in C_2$ and $(z, u) \in C_3$, i.e., realize a (2, 3, 3) – triangle on (y, z). Then y, z, u belong to disjoint C_1-subgraphs. Assume that a 4th C_1-clique exist and choose v in this clique so that $(y, v) \in C_2$. [If necessary realize a (3, 1, 2)-triangle.] Since uC_2yC_2v, u and v are in separate cliques, and (2, 2, 2)-triangles are forbidden, we have $(u, v) \in C_3$. Similarly, $(v, z) \in C_2$ and $(x, v) \in C_3$. Since a (3, 1, 2)-triangle is realizable on (y, u), there is w with $(w, u) \in C_1$ and $(y, w) \in C_3$. Since zC_3yC_3w it follows that $(z, w) \in C_2$ and then a similar argument yields $(w, v) \in C_3$. Now, x and w belong to disjoint C_1-cliques so either $(x, w) \in C_2$ or $(x, w) \in C_3$. But either case produces a forbidden monochrome triangle since xC_2zC_2w and xC_3vC_3w. It follows that $\mathscr{V}$ contains exactly 3 C_1-cliques.

Corollary 5. A^{01}_{14} is chromatic but condition (*) of Theorem 2 fails.

Proof. (second part) Consider the partial scheme $\mathscr{V}'$ below that only uses colors 2 and 3. (Color 2 is represented by a solid line and color 3 by a dashed line.)



Any scheme that contains $\mathscr{V}'$ must have at least 4 C_1-cliques. $\mathscr{V}'$ cannot be extended to a representation of A^{01}_{14} by Proposition 4.

This confirms the conjecture made in [3].

2. Random Color Schemes with Forbidden Triangles

The random schemes considered in this section do not have the disadvantages inherent in the iterative technique. Namely,

(1) there is an easy sufficient condition for a polygroup to be represented by a random scheme, and

(2) the algebra of a random scheme is a double coset algebra.

Random schemes were introduced in [1]. For convenience we take $C = \{1, \ldots, n\}$ and let $\iota(i) = i'$. Random schemes are stronger than those obtained using (*) in the previous section. To make the concept precise we need the following notion. Suppose $\mathscr{V} = \langle V, C_i \rangle_{i \in C}$ is a color scheme and $K \subseteq C$ is a class of triangles. A sequence $F_1, \ldots, F_n$ of subsets of V is *compatible with* K if it is not the case there exist $i, j \in C$, $u \in F_i$, $v \in F_j$ with $(u, v) \in C_k$ where $(i, j', k) \in K$. (Of course $i = j$ is allowed.)

A color scheme $\mathscr{V}$ is a *random n-color scheme with forbidden K* if

(i) no triangle in K is realized in $\mathscr{V}$,

(ii) for every sequence $F_1, \ldots, F_n$ of n finite pairwise disjoint sets of vertices compatible with K, there exist a vertex p such that for every $i \in C$ and $v \in F_i$, $(v, p) \in C_i$ (and, of course, $(p, v) \in C_{i'}$).

Standard arguments show that any two denumerable random n-color schemes with a forbidden K are isomorphic and the automorphism group of such a scheme $\mathscr{V}$ has very strong properties. In [1] it was shown that $\mathfrak{M}_{\mathscr{V}}$ is a double coset algebra $G // G_x$ where $G = \text{Aut}(\mathscr{V})$ and G_x is the stabiliser at a vertex x.

The following result gives a useful condition for showing that a polygroup $\mathfrak{M}$ is represented by a random scheme.

Theorem 6. A random n-color scheme with forbidden K exist if K depends on at most $n - 1$ colors (i.e., one of the n colors does not occur in any triangle of K).

Proof. Do the induction as in Theorem 2 but at the nth stage add only one point for each compatible partition of the partial scheme at that stage. Edges between these new points at the $(n + 1)$-st stage can ee colored with some color missing from K.

It is known that the condition in Theorem 6 is not necessary, only sufficient, for a random scheme to exist. In [1] the condition is applied to show that permutation groups (namely $\text{Aut}(\mathscr{V})$) with certain properties exist. It would be very desirable to know if finite groups with these properties exist. The general problem is to determine for a random n-color scheme $\mathscr{V}$ with a forbidden class K, when there exist a *finite* scheme W with $\mathfrak{M} \cong \mathfrak{M}_{\mathscr{W}}$?

3. Product Constructions

One way to verify that a particular polygroup is chromatic is to show that it can be built up from systems, known to be chromatic, using operations which reserve the property. Thus, it is of interest to determine operations (or constructions) that leave the basic classes of polygroups invariant. In this section two types of products are considered.

Direct products will be considered first. Although the product of two algebras is treated for convenience, it is not difficult to extend the treatment to an arbitrary number of factors. Given polygroups $\mathfrak{A}$ and $\mathfrak{B}$ the *direct Product* $\mathfrak{A} \times \mathfrak{B}$ of $\mathfrak{A}$ and $\mathfrak{B}$ is the system

$$\langle A \times B, *, ^{-1}, (e, e)\rangle$$

where $(a, b)^{-1} = (a^{-1}, b^{-1})$ and

$$(a_0, b_0) * (a_1, b_1) = \{(x, y) : x \in a_0 \,.\, a_1 \text{ and } y \in b_0 \,.\, b_1\}$$

The first result shows that a product of chromatic systems is chromatic.

Suppose $\mathscr{V}_0 = \langle V_0, C_\alpha \rangle_{\alpha \in A - \{e\}}$ and $\mathscr{V}_1 = \langle V_1, C_\beta \rangle_{\beta \in B - \{e\}}$ are color schemes. It is convenient to assume the identity relation occurs among the colors, i.e., $C_e = I_{V_0}$ (in $\mathscr{V}_0$) and $C_e = I_{V_1}$ (in $\mathscr{V}_1$). Consider the product scheme

$$\mathscr{V}_0 \times \mathscr{V}_1 = \langle V_0 \times V_1, C_{(\alpha,\beta)} \rangle \, (\alpha, \beta) \in A \times B \setminus \{(e, e)\}$$

where $C_{(\alpha,\beta)} = \{((a, i), (b, j)) \in (V_0 \times V_1)^2 : (a, b) \in C_\alpha \text{ and } (i, j) \in C_\beta\}$.

Theorem 7. $\mathscr{V}_0 \times \mathscr{V}_1$ is a color scheme and $\mathfrak{M}_{\mathscr{V}_0 \times \mathscr{V}_1} = \mathfrak{M}_{\mathscr{V}_0} \times \mathfrak{M}_{\mathscr{V}_1}$.

Proof. Most of the color scheme properties are trivial. We show (iv). Suppose $((a, i), (b, j))$ belongs to $C_{(\alpha_0,\beta_0)} \cap (C_{(\alpha_1,\beta_1)} \mid C_{(\alpha_2,\beta_2)})$. If $\alpha_k = e$ or $\beta_k = e$ for some $k \in \{0, 1, 2\}$, the condition above degenerates and the conclusion is clear. Assume all α's and β's are $\neq e$. Then $(a, b) \in C_{\alpha_0}$ and $(i, j) \in C_{\beta_0}$ and there exist (c, k) such that

$$(a, c) \in C_{\alpha_1} \quad \text{and} \quad (i, k) \in C_{\beta_1} \quad \text{and} \quad (c, b) \in C_{\alpha_2} \quad \text{and} \quad (k, j) \in C_{\beta_2} \, .$$

Then $C_{\alpha_0} \cap (C_{\alpha_1} \mid C_{\alpha_2}) \neq \emptyset$ and $C_{\beta_0} \cap (C_{\beta_1} \mid C_{\beta_2}) \neq \emptyset$ so $C_{\alpha_0} \subseteq C_{\alpha_1} \mid C_{\alpha_2}$ and $C_{\beta_0} \subseteq C_{\beta_1} \mid C_{\beta_2}$. It follows that $C_{(\alpha_0,\beta_0)} \subseteq C_{(\alpha_1,\beta_1)} \mid C_{(\alpha_2,\beta_2)}$. To see that $\mathfrak{M}_{\mathscr{V}_0 \times \mathscr{V}_1} = \mathfrak{M}_{\mathscr{V}_0} \times \mathfrak{M}_{\mathscr{V}_1}$ notice that allowing $\alpha = e$ and $\beta = e$ simplifies the definition of $*$ in $\mathfrak{M}_{\mathscr{V}_0} \times \mathfrak{M}_{\mathscr{V}_1}$. Namely,

$$(\alpha_0, \beta_0) * (\alpha_1, \beta_1) = \{(\alpha_2, \beta_2) : C_{(\alpha_2,\beta_2)} \cap (C_{(\alpha_0,\beta_0)} \mid C_{(\alpha_1,\beta_1)}) \neq \emptyset\} \, .$$

As in the proof of (iv) above, the product

$$(\alpha_2, \beta_2) \in (\alpha_0, \beta_0) * (\alpha_1, \beta_1)$$

is equivalent to products

$$\alpha_2 \in \alpha_0 \, . \, \alpha_1 \quad \text{and} \quad \beta_2 \in \beta_0 \, . \, \beta_1$$

on the factors. This, in turn, is equivalent to the product in $\mathfrak{M}_{\mathscr{V}_0} \times \mathfrak{M}_{\mathscr{V}_1}$.

What other classes are closed under direct products? For color schemes $\mathscr{V}_0$ and $\mathscr{V}_1$ it is not hard to see that

Lemma 8. $\mathrm{Aut}(\mathscr{V}_0) \times \mathrm{Aut}(\mathscr{V}_1) \cong \mathrm{Aut}(\mathscr{V}_0 \times \mathscr{V}_1)$.

Proof. Consider the map that sends $(\sigma_0, \sigma_1) \in \mathrm{Aut}(V_0) \times \mathrm{Aut}(V_1)$ to the automorphism $[\sigma_0, \sigma_1]$ defined as $[\sigma_0, \sigma_1] \, (a, i) = (\sigma_0(a), \sigma_1(i))$ for all $(a, i) \in V_0 \times V_1$.

Theorem 9. The following classes are closed under direct products:

(1) $Q^2(Group)$

(2) $Q^2_s(Group)$

(3) double coset algebras.

Proof. (1) By 4.1 of [1] $\mathfrak{M} \in Q^2(Group)$ iff $\mathfrak{M} \cong \mathfrak{M}_{\mathscr{V}}$ for some color scheme $\mathscr{V}$ with $\mathrm{Aut}(\mathscr{V})$ transitive on vertices. Given $\mathscr{V}_0, \mathscr{V}_1$ both with such automorphism groups Lemma 8 shows that $\mathrm{Aut}(\mathscr{V}_0 \times \mathscr{V}_1)$ is also transitive on vertices. By Theorem 7, $\mathfrak{M}_{\mathscr{V}_0} \times \mathfrak{M}_{\mathscr{V}_1} \in Q^2(Group)$. The proof of (2) and (3) is similar using 4.2(2) and 4.2 (1)of [1].

We mention one application of direct products. Harrison [6] and Maddox [7] associated a polygroup $\mathfrak{M}(L)$ with each modular lattice $L = \langle L, \vee, \wedge \rangle$ with a minimum element e. Namely, $\mathfrak{M}(L) = \langle L, \cdot, ^{-1}, e\rangle$ where $x^{-1} = x$ and

$$x \,.\, y = \{z \in L : x \vee z = y \vee z = x \vee y\} .$$

It is not hard to check that $\mathfrak{M}(L_1 \times L_2) = \mathfrak{M}(L_1) \times \mathfrak{M}(L_2)$. Theorem 7 implies that whenever the $\mathfrak{M}(L)$ construction associates a chromatic polygroup to lattices L_1 and L_2, the product also gives rise to a chromatic polygroup. A similar conclusion holds for the classes listed in Theorem 9.

We now consider another type of product. Assume that $\mathfrak{A}$ and $\mathfrak{B}$ are polygroups with $A \cap B = \{e\}$ (i.e., only the identity element is common). Define *the extension* of $\mathfrak{A}$ *by* $\mathfrak{B}$

$$\mathfrak{A}[\mathfrak{B}] = \langle M, *, ^I, e\rangle$$

where $M = (A \setminus \{e\}) \cup (B \setminus \{e\}) \cup \{e\}$, $e^I = e$, $x^I = x^{-I}$ (whichever operation $^{-1}$ is appropriate for x).

$$e * x = x * e = x ,$$

and

$$x * y = \begin{cases} x \,.\, y & \text{if} \quad x, y \in A \\ x & \text{if} \quad x \in B,\ y \in A \\ y & \text{if} \quad x \in A,\ y \in B \\ x \,.\, y & \text{if} \quad x, y \in B,\ y \neq x^{-1} \\ x \,.\, y \cup A & \text{if} \quad x, y \in B,\ y = x^{-1} \end{cases}$$

Several examples are worked out in [4]. Other suitable names for $\mathfrak{A}[\mathfrak{B}]$ include the "wreath product" of $\mathfrak{A}$ by $\mathfrak{B}$ (due to the way a double coset representation for $\mathfrak{A}[\mathfrak{B}]$ is constructed from such representations of $\mathfrak{A}$ and $\mathfrak{B}$) and the "composition" of $\mathfrak{A}$ with $\mathfrak{B}$ (due to the way a color scheme for $\mathfrak{A}[\mathfrak{B}]$ is constructed from such representations for $\mathfrak{A}$ and $\mathfrak{B}$).

Theorem 10. The following classes are closed under the extension operations $\mathfrak{A}[\mathfrak{B}]$.

(a) Chromatic
(b) $Q^2(Group)$
(c) $Q_s^2(Group)$
(d) double coset algebras

Proof. The idea for the proof of (a) is to take a color scheme $\mathscr{V}$ that represents $\mathfrak{B}$ and replace each vertex of $\mathscr{V}$ by a copy of a color scheme $\mathscr{W}$ that represents $\mathfrak{A}$. The resulting scheme represents $\mathfrak{A}[\mathfrak{B}]$. If both $\mathscr{V}$ and $\mathscr{W}$ have transitive automorphism groups, it is not hard to see that the scheme representing $\mathfrak{A}[\mathfrak{B}]$ also has one. Thus, $Q^2(Group)$ is closed under $\mathfrak{A}[\mathfrak{B}]$. (b) and (c) have similar proofs. Different proofs are given in [4].

We conclude with two examples of how to use $\mathfrak{A}[\mathfrak{B}]$. Recall that the ordered sum $L_0 \oplus L_1$ of two bounded lattices is the lattice obtained by identifying the minimum element of L_1 with the maximal element L_0.

Proposition 11. For bounded modular lattices L_0 and L_1, $\mathfrak{M}(L_0 \oplus L_1) = \mathfrak{M}(L_0)[\mathfrak{M}(L_1)]$. Thus, $L_0 \oplus L_1$ yields a chromatic polygroup whenever L_0 and L_1 do.

Composition series play an important role in the study of groups. Polygroups (and in general multigroups) also exhibit similar series (c.f., [5]). We outline the notion of an *ultragroup* from Roth [8]. The core of a polygroup $\mathfrak{M}$, written Core(M), is the subpolygroup generated by $\bigcup\{a \,.\, a^{-1} : a \in M\}$. For a subpolygroup $\mathfrak{N}$ of $\mathfrak{M}$ introduce a full conjugation $\Theta_{\mathfrak{N}}$ on $\mathfrak{M}$ by

$$a\Theta_{\mathfrak{N}}b \quad \text{iff} \quad b \in NaN$$

$\mathfrak{M} \,//\, \Theta_{\mathfrak{N}}$ is a polygroup. An *ultragroup* is a polygroup $\mathfrak{M}$ for which there exist a chain of subpolygroups

$$\{e\} = \mathfrak{N}_k \subseteq \mathfrak{N}_{k-1} \subseteq \ldots \subseteq \mathfrak{N}_0 = \mathfrak{M}$$

where $\mathfrak{N}_i \,//\, \Theta_{\mathfrak{N}_{i-1}}$ is a group for all $i < k$. The groups $\mathfrak{N}_0 \,//\, \Theta_{\mathfrak{N}_1}, \ldots$ are called the *factors* of the series. Note that Core($\mathfrak{A}[\mathfrak{B}]$) $= A \cup$ Core($\mathfrak{B}$) so if $\mathfrak{B}$ is a group, Core($\mathfrak{A}[\mathfrak{B}]$) $= A$.

Proposition 12. Given groups $G_0, G_1, \ldots, G_{k+1}$ and $0 < i < k$, let $\mathfrak{N}_i$ denote the extension $(\ldots(G_0[G_1])\ldots)[G_{i-1}]$. Then

(1) $\mathfrak{N}_k$ is an ultragroup with factors $G_{k-1}, \ldots, G_0$.

(2) $\mathfrak{N}_k$ is a double coset algebra.

Proof. (1) Since $\mathfrak{N}_i = \mathfrak{N}_{i-1}[G_{i-1}]$ and Core($\mathfrak{N}_i$) $= \mathfrak{N}_{i-1}$, $N_k \supset \mathfrak{N}_{k-1} \supset \ldots \ldots \supset \mathfrak{N}_1 \supset \{e\}$ gives the lower ultra-series for $\mathfrak{N}_k$. Moreover, note that for $a, b \in N_i$,

$$a\Theta_{\mathfrak{N}_{i-1}}b \quad \text{iff} \quad a, b \in N_{i-1} \quad \text{or} \quad a = b\,.$$

Thus, $\mathfrak{N}_i \,//\, \Theta_{\mathfrak{N}_{i-1}} = \mathfrak{N}_i \,//\, \mathfrak{N}_{i-1} = G_{i-1}$.

(2) follows from Theorem 10.

It would be interesting to know whether every ultragroup is a double coset algebra.

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