COLOR SCHEMES FORBIDDING MONOCHROME TRIANGLES**

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The purpose of this note is to raise the question of the existence of certain edge colorings and to contribute a few simple remarks. A lower estimate r for the Ramsey number $N_n(3,2)$ is obtained by coloring the edges of the complete graph K_r with n colors in such a way that monochrome triangles do not appear. The author is interested in edge colorings in which monochrome triangles are the only colored triangles forbidden. Moreover, it is desired that the coloring possess certain regularity properties. For the motivation for these requirements the reader should see [1].

1. PRELIMINARIES

For convenience colors will be denoted by positive integers. A triple (i,j,k) of colors is called a <u>(colored)</u> <u>triangle</u>. Given a set $\{1,\ldots,n\}$ of n colors a <u>symmetric</u> n-<u>color</u> <u>scheme</u> is an edge coloring $\mathscr{S} = (V,C_1,\ldots,C_n)$ of a complete (undirected) graph with vertex set V such that

(i) each color is present on some edge eminating from each vertex,

(ii) for colors i,j and k if there exist $a,b,c \in V$ with $(a,c) \in C_i$, $(c,b) \in C_j$ and $(a,c) \in C_k$, then, for every $x,y \in V$ with $(x,y) \in C_k$ there exist $z \in V$ such that $(x,z) \in C_i$ and $(z,y) \in C_j$.

Condition (ii) says that if V contains one triangle with edges colored (i,j,k), then on every edge colored k it is possible to realize an (i,j,k) triangle. In precise set-theoretic terms, using | to denote relation composition, (ii) means:

(ii') for colors i,j,k, $C_k \cap (C_i | C_j) \neq \emptyset$ implies $C_k \subseteq C_i | C_j$.

A color scheme $\mathscr{S}$ <u>forbids</u> a triangle (i,j,k) if $C_k \cap (C_i | C_j) \neq \emptyset$. The <u>forbidden</u> <u>class</u> $FC(\mathscr{S})$ of $\mathscr{S}$ is the class of triangles forbid by $\mathscr{S}$.

*Research was supported, in part, by NSF grant no. MCS-8003896 and by the Citadel Development Foundation.

**This paper is in final form and will not appear elsewhere.

CONGRESSUS NUMERANTIUM, VOL. 39 (1983), pp. 231-236

Association schemes form an important class of symmetric color schemes. For an n-class association scheme with parameters n_i and p_{ij}^k, the scheme has a property even stronger than (ii). Namely, for every $x,y \in V$ with $(x,y) \in C_k$

$$p_{ij}^k = |\{ z \in V : xC_i zC_j y \}|$$

is independent of the choice of x and y. In an association scheme, a triangle (i,j,k) is forbidden if, and only if, $p_{ij}^k = 0$.

Problem 1. For which positive integers n does there exist a symmetric n-color scheme $\mathscr{S}$ with $FC(\mathscr{S}) = \{(1,1,1),\ldots,(n,n,n)\}$?

Two stronger forms of this question are also of interest. For $n \geq 1$ let r(n) denote the maximal size of V for which there exist a symmertic n-color scheme on V forbidding exactly the monochrome triangles. Set r(n) = 0 if no such scheme exist.

Problem 2. What is r(n) ?

What is the situation for association schemes?

Problem 3. For which positive integers n does there exist an n-class association scheme such that $p_{ij}^k > 0$ iff $|\{i,j,k\}| > 1$?

2. CYCLOTOMIC SCHEMES

Some techniques for constructing symmetric color schemes are given in [1] and [2]. Most of the techniques used to give lower bounds for Ramsey numbers do not build in enough repetition to satisfy condition (ii) for the schemes we seek. An exception to this are the cyclotomic schemes used in [4] to produce sum-free sets in finite fields. We examine these schemes below.

In the remainder of this paper we assume that p is a prime number and m,n and k are positive integers such that $p^k = 1 + mn$. Choose x such that $\langle x\rangle = GF(p^k)^*$, i.e., x generates the multiplicative group of $GF(p^k)$. The cosets of the subgroup $H = \langle x^n\rangle$ of $GF(p^k)^*$ are called cyclotomic classes. Namely, the ith cyclotomic class H_i is

$$H_i = x^i H = \{ x^{rn+i} : r < m \}.$$

The classes can be used to introduce an edge coloring of the complete graph with vertex set $GF(p^k)$. For a color $i \in \{1,\dots,n\}$ an edge (a,b) has color i (i.e., $(a,b) \in C_i$) iff $b-a \in H_{i-1}$.

The theorem below says that under certain conditions $(GF(p^k), C_1,\dots,C_n)$ is an n-class association scheme. This scheme is called a cyclotomic (n-color) scheme. The 3-coloring of the complete graph on $GF(16)$ given by Greenwood and Gleason [3] is cyclotomic.

For $i,j < n$ define $s(i,j)$ to be the number of solutions to the equation $y+1=z$ where $y \in H_i$ and $z \in H_j$.

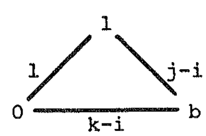
Theorem 1. For $p=2$ or m an even integer, the system $(GF(p^k), C_1,\dots,C_n)$ defined above is an n-class association scheme with parameters

$$p^0_{ij} = m\delta_{ij}, \quad p^k_{i0} = \delta_{ik} \quad \text{and} \quad p^k_{ij} = s(j-i,k-i).$$

Proof. The assumption that either $p=2$ or m is an even integer guarantees that $-H = H$ so the coloring is well defined on the undirected graph. By translation every triangle in $GF(p^k)$ with vertices a,b and c corresponds to one, having the same color pattern, with the a-vertex at the origin. Color patterns may also be cycled by multiplying (each vertex of) the triangle by a field element. In this way it can be seen that for $(a,b) \in C_k$

$$|\{c \in V : aC_i cC_j b\}|$$

is independent of a and b and, in fact, the number is the same as the number of triangles of the form



which is $s(j-i,k-i)$.

Remark. The theorem above is essentially due to Street and Whitehead [4] who considered cyclotomic schemes on $GF(p^{2r})$ where $p^r \equiv 1 \pmod 3$, $n = p^r-1$ and $m = p^r+1$. In this case the cyclotomic classes H_i are sum-free sets and the schemes forbid monochrome triangles. In general, however, from the description of the p^k_{ij}'s it can be seen that these schemes forbid many types of triangles, not just monochrome triangles.

For example, for $p=7$, $r=1$, $n=6$ and $m=8$ the 6-color scheme on $GF(49)$ forbids the following type of triangles:

$$(i,i,i),\ (i,i,i+5),\ (i,i+1,i+3).$$

[$i \in \{1,\dots,6\}$ and addition is computed mod 6.]

Cyclotomic schemes do allow us to produce the desired association schemes for small n and to compute $r(n)$ for a few values. Before giving these results it is useful to consider the extent to which a cyclotomic scheme depends on the generator of the multiplicative group.

<u>Theorem 2</u>. Suppose x and x^s both generate $GF(p^k)^*$ and that $H_0,\dots,H_{n-1}$ (respectively, $H'_0,\dots,H'_{n-1}$) are the cyclotomic classes in $GF(p^k)$ determined by x (respectively, x^s). Then $H_0 = H'_0$ and the classes $H'_1,\dots,H'_{n-1}$ are the same as $H_1,\dots,H_{n-1}$ up to a rearrangement of their order.

<u>Proof</u>. The $\gcd(p^k-1,s)=1$ so the map that sends x^i to x^{is} is an automorphism of $GF(p^k)^*$ that carries $H_0 = \langle x\rangle$ onto $H'_0 = \langle x^{sn}\rangle$. Since $H'_0 \subseteq H_0$, $H_0 = H'_0$. It follows that the other cyclotomic classes determined by x and x^s are the same (in some order) since they are the cosets of $H_0 = H'_0$.

<u>Corollary 3</u>. If $\mathscr{L}$ and $\mathscr{D}$ are two cyclotomic n-color schemes on a finite field and if K is a class of colored triangles that is invariant under all permutations of colors, then

(1) $\mathscr{L}$ and $\mathscr{D}$ differ only by a permutation of colors,

(2) $|FC(\mathscr{L})| = |FC(\mathscr{D})|$,

(3) $FC(\mathscr{L}) = K$ if and only if $FC(\mathscr{D}) = K$.

<u>Theorem 4</u>. An n-class association scheme with forbidden class $\{(1,1,1),\dots,(n,n,n)\}$ exist for $n < 6$. Moreover, $r(1)=2$, $r(2)=5$, $r(3)=16$, $r(4) \geqslant 41$, and $r(5) \geqslant 101$.

<u>Proof</u>. TABLE 1 below shows that $r(n)$ is at least the asserted value. Equality in the first three cases follows since the Ramsey numbers are known for these n.

For n colors, n < 7, TABLE 1 lists the forbidden class of cyclotomic schemes on select fields $V = GF(p^k)$. In view of Corollary 3(3), in most cases, the forbidden class does not depend on the element used to determine the scheme. In the cases where the forbidden class does depend on the generator of V* it is clear from 3(2) that none of the permuted cousins will forbid exactly the class of monochrome triangles. It should be emphasized that in cases where the cyclotomic scheme has a empty forbidden class, such as the 4-color scheme on GF(49), it may be possible to find a, necessarily, non-cyclotomic color scheme on the same set which forbids exactly the monochrome triangles.

TABLE 1

n = #colors	$p^k = 1+nm$	Field	Forbidden class
2	5 = 1+2.2	GF(5)	(i,i,i)
	9 = 1+2.4	$GF(3^2)$	∅
3	13 = 1+3.4	GF(13)	(i,i,i)
	16 = 1+3.5	$GF(2^4)$	(i,i,i)
4	41 = 1+4.10	GF(41)	(i,i,i)
	49 = 1+4.12	$GF(7^2)$	∅
5	11 = 1+5.2	GF(11)	(i,i,i) + others
	31 = 1+5.6	GF(31)	(i,i,i+3),(i,i,i+4)
	71 = 1+5.14	GF(71)	(i,i,i)
	101 = 1+5.20	GF(101)	(i,i,i)
	121 = 1+5.24	$GF(11^2)$	∅
6	49 = 1+6.8	$GF(7^2)$	(i,i,i) + others
	61 = 1+6.10	GF(61)	(i,i,i),(i,i,i+4)
	109 = 1+6.18	GF(109)	(i,i,i+1)
	121 = 1+6.20	$GF(11^2)$	∅
	169 = 1+6.28	$GF(13^2)$	∅

For a fixed number of colors it is interesting to note the change in the size of the forbidden class as the field size increases. Is it true for symmetric n-color schemes $\mathscr{C}$ and $\mathscr{D}$ that $|FC(\mathscr{C})| \leqslant |FC(\mathscr{D})|$ whenever $|\mathscr{C}| \leqslant |\mathscr{D}|$?

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