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Polygroups Derived from Cogroups

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A polygroup is a completely regular, reversible in itself multigroup in the sense of Dresher and Ore [4]. These systems occur naturally in the study of algebraic logic [1, 2]. Some algebraic and combinatorial properties were developed in [3]. The question of whether or not Q^2(Group), the class of double quotients of groups, is the same as the class of all chromatic polygroups is important for many reasons. In this paper double quotients of groups are constructed in a natural way from cogroups. Cogroups were introduced by Eaton [5] in an attempt to axiomatize D-hypergroups, i.e., systems obtainable from groups by right coset decompositions with respect to, not necessarily normal, subgroups. Eaton's axioms apply only to finite systems. Utumi [7] formulated a general notion and gave an example of a cogroup not isomorphic to a D-hypergroup. Utumi's scalar partition hypergroups appear in Section 2. The main results, in Section 3, show that polygroups derivable from cogroups are chromatic and they include all double quotients of groups.

1. PRELIMINARIES

We recall the following definition from [3].

A *polygroup* is a system $\mathfrak{M} = \langle M, \cdot, e, ^{-1} \rangle$ where $e \in M$, $^{-1}$ is a unary operation on M, $\cdot$ maps M^2 into nonempty subsets of M, and the following axioms hold for all x, y, z in M:

(P_1) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,

(P_2) $e \cdot x = x = x \cdot e$,

(P_3) $x \in y \cdot z$ implies $y \in x \cdot z^{-1}$ and $z \in y^{-1}x$.

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When discussing multivalued systems it is convenient to identify elements with singletons and to set $A \cdot B = \bigcup \{a \cdot b: a \in A, b \in B\}$ whenever A and B are subsets of M. The following elementary facts about polygroups follow easily from the axioms: $e \in x \cdot x^{-1}$, $e \in x^{-1} \cdot x$, $e^{-1} = e$, and $(x^{-1})^{-1} = x$.

An important class of polygroups is derived from color schemes, a notion that extends D. G. Higman's homogeneous coherent configuration (see [6]). In the defintion of a color scheme presented below the relative product (or composition) of two relations is denoted by $|$ and the inverse (or converse) of a relation is denoted by $^{\cup}$. Suppose C is a set (of colors) and ε is an involution of C. A *color scheme* is a system $\mathscr{V} = \langle V, C_x \rangle_{x \in C}$ where

(i) $\{C_x: x \in C\}$ partitions $V^2 - I = \{(a, b) \in V^2: a \neq b\}$;

(ii) $C_x^{\cup} = C_{\varepsilon(x)}$ for all $x \in C$;

(iii) for all $x \in C$, $a \in V$ there exist $b \in V$, $(a, b) \in C_x$;

(iv) $C_x \cap (C_y \mid C_z) \neq 0$ implies $C_x \subseteq C_y \mid C_z$, i.e., the existence of a path colored (y, z) between two vertices joined by an edge colored x is independent of the two vertices.

Given a color scheme $\mathscr{V}$, choose a new symbol $I \notin C$. (Think of I as the identity relation on V.) The *algebra* (*color algebra*) of $\mathscr{V}$ is the system $\mathfrak{M}_{\mathscr{V}} = \langle C \cup \{I\}, \cdot, I, ^{-1} \rangle$ where $x^{-1} = \varepsilon(x)$ for $x \in C$, $I^{-1} = I$,

$$x \cdot I = x = I \cdot x \qquad \text{for all} \quad x \in C \cup \{I\},$$

and, for $x, y, z \in C$,

$$x \cdot y = \{z \in C: C_z \subseteq C_x \mid C_y\} \cup \{I: y = x^{-1}\}.$$

The algebra $\mathfrak{M}_{\mathscr{V}}$ is a polygroup. A polygroup is called *chromatic* if it is isomorphic to an algebra $\mathfrak{M}_{\mathscr{V}}$ for some color scheme $\mathscr{V}$.

A natural example of a chromatic polygroup is the system $G//H$ of all double cosets of a group G modulo a subgroup H. Namely, the algebra

$$G//H = \langle \{HgH: g \in G\}, \cdot, H, ^{-1} \rangle$$

where $(HgH) \cdot (Hg'H) = \{Hghg'H: h \in H\}$ and $(HgH)^{-1} = Hg^{-1}H$. To see that $G//H$ is chromatic consider the color scheme $\mathscr{V} = \langle V, C_x \rangle_{x \in C}$, where $C = (G//H) \backslash \{H\}$, $V = \{Ha: a \in G\}$ and for $x \in C$, $C_x = \{(Ha, Hb): ab^{-1} \in x\}$. It is not hard to verify that $G//H$ is isomorphic to $\mathfrak{M}_{\mathscr{V}}$.

The (double) quotient $G//\theta$ of a group by a full conjugation (defined below) generalizes the double coset construction above. For a subgroup H of a group G the equivalence relation θ on G defined by $x\theta y$ iff $HxH = HyH$ is a full conjugation in the sense below and $G//\theta = G//H$. An equivalence relation θ on a polygroup $\mathfrak{M}$ is called a (*full*) *conjugation* on $\mathfrak{M}$ if

(i) $x\theta y$ implies $x^{-1}\theta y^{-1}$,

(ii) $z \in x \cdot y$ and $z'\theta z$ implies $z' \in x' \cdot y'$ for some $x'\theta x$ and $y'\theta y$.

The collection of all θ-classes, with operations induced from $\mathfrak{M}$, forms a polygroup denoted by $\mathfrak{M}//\theta$. The class of all polygroups isomorphic to $G//\theta$, where θ is a full conjugation on a group G, is denoted Q^2(Group). In [3] it is shown that Q^2(Group) is a class of chromatic polygroups. Conjugations are exactly the equivalence relations that occur as kernels of homomorphisms in the sense of [2].

A (full) conjugation θ that satisfies the additional property

(iii) $x\theta e$ implies $x = e$

is called a *special conjugation*.

The definition of a cogroup given below is equivalent to the one formulated by Utumi [7]. The cardinality axiom assumed by both Utumi and Eaton does not play a role in our development. The apparently weaker notion of weak-cogroup is obtained by removing this assumption.

A *weak cogroup* is a system $\langle A, \cdot, ^{-1}, e\rangle$, where $e \in A$; $x \cdot y$ is a nonempty subset of A for $x, y \in A$; x^{-1} is a nonempty subset of A for all $x \in A$; and the following axioms hold for all $x, y, z \in A$:

(C_1) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,

(C_2) $e \cdot x = x$,

(C_3) $y \in x^{-1} \leftrightarrow e \in x \cdot y$,

(C_4) $x \in y \cdot z \rightarrow y \in x \cdot z^{-1}$ and $z \in y^{-1} \cdot x$,

(C_5) $x \cdot y \cap z \cdot y \neq 0 \rightarrow x \in z \cdot e$.

A weak cogroup is called a *cogroup* if, in addition, it satisfies the axiom

(C_6) $|x \cdot y| = |x \cdot z|$ for all $x, y, z \in A$.

If H is a subgroup of a group G, the system $G/H = \langle\{Hg\colon g \in G\}, \cdot, ^{-1}, H\rangle$ of all right cosets becomes a cogroup using the operations $(Hg) \cdot (Hk) = \{Hghk\colon h \in H\}$ and $(Hg)^{-1} = \{Hg^{-1}h\colon h \in H\}$. The system G/H is known as a *D-hypergroup*. Not every cogroup is a D-hypergroup (Utumi [7]).

Elements x and y in a weak cogroup are called *e-conjugates*, in symbols $x \approx y$, iff $x \in y \cdot e$. It is easy to see that $\approx$ is an equivalence relation on A, the $\approx$-class of e is $\{e\}$, and

$$x \in y \cdot e \quad \text{iff} \quad xe = ye \quad \text{iff} \quad xz = yz \quad \text{for all } z.$$

The product xe is the $\approx$-class that contains x.

2. Derived Polygroups

The canonical example of a cogroup is G/H while that of a polygroup is $G//H$. Every element in $G//H$ is a $\approx$-class of G/H. This suggests a way to construct polygroups from arbitrary weak cogroups.

LEMMA 1. *Suppose A is a weak cogroup and $\approx$ is the relation of e-conjugation. Then*

(1) $(ae)(be) = (ab)e = \{ce: c \in ab\}$ and $(ae)^{-1} = a^{-1}e = a^{-1}$.

(2) *The system $A/\approx$ of all $\approx$-classes, with operations inherited from A, is a polygroup.*

Proof. Straightforward. As an example consider $a^{-1}e \subseteq a^{-1}$. Suppose $b \in ce$ for some $c \in a^{-1}$. Then $e \in ac$, which implies $e \in ca = ba$. Hence $e \in ab$ and so $b \in a^{-1}$.

The system $A/\approx$ is called the *polygroup derived from A*.

The following definition and lemma are essentially due to Utumi [7]. For an equivalence relation θ on a weak cogroup $\langle A, \cdot, {}^{-1}, e\rangle$, let

$$A^{*\theta} = \langle A, *, {}^{-1^*}, e^*\rangle$$

where $e^* = e$, $x * y = (\theta x) \cdot y$, and $x^{-1^*} = (\theta x)^{-1}$; i.e., $x^{-1^*} = \bigcup \{y^{-1}: y \in \theta x\}$. $A^{*\theta}$ is called a scalar partition hypergroupoid with respect to θ. We refer to the structure as A^* whenever θ is understood.

LEMMA 2. *Suppose A is a weak cogroup and θ is an equivalence relation on A with $x \cdot e \subseteq \theta x$ for all x. Then A^* is a weak cogroup iff*

(i) $\theta e = \{e\}$,

(ii) $\theta x^{-1} = (\theta x)^{-1}$,

(iii) $\theta((\theta x)y) = (\theta x)(\theta y)$.

Moreover, if A is cogroup, so is A^*.

The condition $x \cdot e \subseteq \theta x$ in Lemma 2 is only needed to establish the implication $\rightarrow$.

For either a weak cogroup A or a polygroup $\mathfrak{A}$, an equivalence relation θ on A that satisfies conditions (i), (ii), (iii) in Lemma 2 and has $x \cdot e \subseteq \theta x$ for all $x \in A$ is called an *Utumi partition*. The condition, $x \cdot e \subseteq \theta x$ for all $x \in A$, is redundant when $\mathfrak{A}$ is a polygroup. These partitions are closely related to special conjugations.

LEMMA 3. *Suppose $\mathfrak{M}$ is polygroup. Then*

(1) *θ is a full conjugation of $\mathfrak{M}$ iff*

(i) $(\theta x)^{-1} = \theta x^{-1}$, *and*

(ii) $\theta((\theta x)y) = (\theta x)(\theta y)$.

(2) *θ is a special conjugation in $\mathfrak{M}$ iff θ is a Utumi partition.*

Proof. (1) Assume θ is a full conjugation. Condition (i) of the definition easily implies (i) of the lemma. Condition (ii) of the definition of conjugation implies that $(\theta x)(\theta y)$ is a union of θ-classes so $\theta((\theta x)y) \subseteq (\theta x)(\theta y)$. Now assume $z \in x' \cdot y'$, $x'\theta x$, and $y'\theta y$. By (P_3), $y' \in x'^{-1} \cdot z$ so $y \in x'' \cdot z$, where $x''\theta x'^{-1}$ and $z'\theta z$ by (ii) of the conjugation definition. By (P_3) again and (i),

$$z\theta z' \in (\theta x) \cdot y$$

which shows equality in (ii). A similar argument established that a θ with properties (i) and (ii) is a full conjugation. Part (2) is obvious.

COROLLARY. *If θ is a special conjugation on a group G, then $G^{*\theta}$ is a cogroup and $G^{*\theta}/\approx = G//\theta$.*

Proof. The equality of quotients follows because $x \approx y$ iff $x \in y^*e = \theta y$ iff $\theta x = \theta y$.

This corollary will be generalized to full conjugations in the next section.

3. THE RELATIONSHIP TO DOUBLE QUOTIENTS

Suppose H is a subgroup of a group G and π is a Utumi partition on G/H. Let

$$G[H, \pi] = (G/H)^{*\pi}/\approx.$$

Also, define $\bar{\pi}$ on G^2 by

$$g_1 \bar{\pi} g_2 \leftrightarrow (\pi(Hg_1))H = (\pi(Hg_2))H.$$

For readability, $(\pi(A))B$ is written as $\pi(A)B$.

Lemmas 1 and 2 show that $G[H, \pi]$ is a polygroup. Also, in $(G/H)^*$, $Hg_1 \approx Hg_2$ iff $\pi(Hg_1)H = \pi(Hg_2)H$. Thus, $\pi(Hg)H$ is the $\approx$-class of Hg. Also, $\pi(Hg)H = \pi(Hg)$ in $(G/H)^*$ since a Utumi partition satisfies $x \cdot e \subseteq \pi x$. Thus, $\approx$ coincides with π.

LEMMA 4. *$\bar{\pi}$ is a full conjugation on G and*

$$G[H, \pi] \cong G//\bar{\pi}.$$

Proof. Clearly $\bar{\pi}$ is an equivalence relation on G. To check condition (i) for a conjugation suppose $x\bar{\pi}y$. Then, $Hx \approx Hy$ in $(G/H)^*$. Since Hx^{-1} is an inverse of Hx, e-conjugate elements have the same inverses, and every two inverses of an element are e-conjugate, $Hx^{-1} \approx Hy^{-1}$. Thus, $x^{-1}\bar{\pi}y^{-1}$ and (i) holds.

To check condition (ii) for $\bar{\pi}$ to be a conjugation assume $x'\bar{\pi}x$ and $x = y \cdot z$. Then $Hx' \in \pi(Hx)$ and $Hx \in (Hy)(Hz)$; so, in G/H,

$$\begin{aligned} Hx' \in \pi(Hx) &\subseteq \pi((Hy)(Hz)) \\ &\subseteq \pi(\pi(Hy)\, Hz) \\ &= \pi(Hy)\, \pi(Hz) \end{aligned}$$

by the Utumi property. Thus, $Hx' \subseteq (Hy'')(Hz'')$ for some $Hy'' \in \pi(Hy)$ and $Hz'' \in \pi(Hy)$. Then $x' = y' \cdot z'$ for some $y' \in Hy''$ and $z' \in Hz''$. Since $\pi(Hy') = \pi(Hy'') = \pi(Hy)$, $y'\bar{\pi}y$. Similarly $z'\bar{\pi}z$. Thus (ii) holds and $\bar{\pi}$ is a conjugation on G.

The correspondence that sends $\bar{\pi}g$ to $Hg/\!\approx\, = \pi(Hg)H = \pi(Hg)$ is clearly a bijection between polygroups $G//\bar{\pi}$ and $G[H, \pi]$. The identity elements correspond as well as inverses since $(\bar{\pi}x)^{-1} = \pi x^{-1}$ and $(\pi(Hx))^{-1} = \pi(Hx^{-1})$. To see that the correspondence is an isomorphism observed that $\pi(Hx) \in \pi(Hy) \cdot \pi(Hz)$ is equivalent to $\bar{\pi}(x) \in \bar{\pi}(y) \cdot \bar{\pi}(z)$.

The lemma above shows that polygroups derived from Utumi partitions on D-hypergroups are double quotients of groups. The next result establishes the converse. Namely, every double quotient of a group is derivable from a D-hypergroup with a Utumi partition.

THEOREM 5. *For any $G//\theta \in Q^2$(Group) there exist a subgroup H of G and a Utumi partition π on G/H such that*

$$G//\theta \cong G[H, \pi].$$

Proof. Given G and θ, $H = \theta e$ is a subgroup of G. Define π on G/H by

$$(Hg_1)\, \pi(Hg_2) \leftrightarrow g_1 \theta g_2.$$

Since θ is a full conjugation on G, $Hg_1 = Hg_2$ implies $g_1\theta g_2$, from which it follows that π is well defined (i.e., it factors through the quotient mod H). Clearly π is an equivalence relation on G/H and, since a conjugation θ has the property $g\theta gh$ for all $h \in H$, $(Hg)H \subseteq \pi(Hg)$. We claim

(1) π is a Utumi partition on G/H.

Since $H = \theta e$, $(Hg)\,\pi H$ easily implies $Hg = H$. It remains to check (ii) and (iii) of Lemma 2.

(ii) $(\pi(Hg))^{-1} = \pi(Hg^{-1})$

Suppose $Hg_0 \in (\pi(Hg))^{-1}$. Then $H \in Hg_0 Hg'$ for some $g'\theta g$ which yields $g'^{-1}\theta g_0$. Since θ is a conjugation, $g'^{-1}\theta g^{-1}$, from which we obtain $Hg_0 \in \pi(Hg^{-1})$. The converse is similar.

Now consider

(iii) $\pi((\pi(Hg_1))\,Hg_2) = \pi(Hg_1)\,\pi(Hg_2)$.

To establish $\subseteq$ assume $(Hg)\,\pi(Hg') \in (\pi(Hg_1))(Hg_2)$ for some $g'\theta g$. It follows that $g' = hg_1'h'g_2 = g''g_2$ for some h, $h' \in H$ and $g_1'\theta g_1$, where $g'' = hg_1'h'$. Since θ is a conjugation, it follows that $g = g_3\,g_2'$ for some $g_3\theta g''\theta g_1'\theta g_1$ and $g_2'\theta g_2$. Thus, $Hg = Hg_3 \cdot g_2' \in \pi(Hg_1) \cdot \pi(Hg_2)$ as desired. A similar argument yields $\supseteq$.

This completes the proof of (1). It follows from (1), Lemma 2, and Lemma 1 that

(2) $(G/H)^{*\pi}$ is a cogroup and $G[H, \pi]$ is a polygroup.

In $(G/H)^{*\pi}$,

$$\begin{aligned} Hg_1 \approx HG_2 \quad &\text{iff} \quad Hg_1 \in \pi(Hg_2)H \\ &\text{iff} \quad Hg_1 = Hg_2'h \end{aligned}$$

for some $g_2'\theta g_2$ and $h \in H$. Since θ is a conjugation and $H = \theta e$, $g_2'h\theta g_2'$, which implies $g_1\theta g_2$. On the other hand $g_1\theta g_2$ implies $(Hg_1)\,\pi(Hg)$, which yields $Hg_1 \approx Hg_2$. Thus, we obtain

(3) $Hg_1 \approx Hg_2$ iff $g_1\theta g_2$.

By (3), $\theta = \bar{\pi}$ (introduced in Lemma 4), so Lemma 4 yields

$$G/\!/\theta \cong G[H, \pi]$$

which completes the proof of Theorem 5.

As noted before the previous two results yield a factorization of double quotients.

COROLLARY. $\mathfrak{M} \in Q^2(\text{Group})$ *iff* $\mathfrak{M} \cong G[H, \pi]$ *for some subgroup H of a group G and Utumi partition π on G/H.*

Denote the class of all polygroups derived from weak cogroups (resp., cogroups) by the construction in Lemma 1 as D(w-cogroup) (resp., D(cogroup)). Then Theorem 5 says

COROLLARY. $Q^2(\text{Group}) \subseteq D(\text{cogroup})$.

We conclude by observing that all polygroups derived from weak cogroups are chromatic, extending Theorem 2.3 in [3].

THEOREM 6. *For every weak cogroup A, $\mathfrak{M} = A/\approx$ is chromatic.*

Proof. Let $C = \{X \in M : X \neq \{e\}\}$ and for $X \in C$ let

$$C_X = \{(a, b) \in A^2 : a \in Xb\}.$$

It is not difficult to show that $\mathscr{V} = \langle A, C_x \rangle_{x \in C}$ is a color scheme. As a sample of the argument consider

(iv) $C_X \cap (C_Y \mid C_Z) \neq 0$ implies $C_X \subseteq C_Y \mid C_Z$.

Suppose $a \in Xb$, $a \in Yc$ and $c \in Zb$ for some $c \in A$. Then $a \in YZb$, from which it follows that $a \in ub$ for some $u \in y \cdot z$, where $y \in Y$ and $z \in Z$. Also, $a \in xb$ for some $x \in X$. Thus, axiom (C_5) yields $u \approx x$ and therefore $X \subseteq YZ$. Now, for any $(r, s) \in C_X$, $r \in Xs \subseteq YZs$, which gives $(r, s) \in C_Y \mid C_Z$ as desired.

The other conditions are verified in a similar way. In particular, the argument used to show (iv) also shows that for X, Y, $Z \neq \{e\}$, $X \in Y \cdot Z$ (in $\mathfrak{M}_{\mathscr{V}}$) iff $C_X \subseteq C_Y \mid C_Z$ iff $X \subseteq YZ$ (in $\mathfrak{M}$). This is the key step in verifying the natural map from $\mathfrak{M}_{\mathscr{V}}$ to $\mathfrak{M}$ is an isomorphism.

4. CONCLUDING REMARKS

The results above raise several natural problems.

1. Which of the following inclusions are proper?

$$Q^2(\text{Group}) \subseteq D(\text{cogroup}) \subseteq D(w\text{-cogroup}) \subseteq \text{Chromatic}.$$

2. Is every cogroup isomorphic to a scalar partition hypergroup of a coset system? (That is, $\cong (G/H)^{*\pi}$ for some groups G, H and Utumi partition π?)

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