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Combinatorial aspects of relations

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Abstract. A multivalued algebra called a polygroup is introduced and used to establish connections among relation algebras, permutation groups, and edge-colored graphs.

1. Introduction

This paper deals with certain algebraic systems that are closely related to structures that occur in algebraic logic (relation algebras, cylindric algebras), to group complexes, and to combinatorial properties of regular edge-colored graphs. One goal of the paper is to illustrate how these areas enrich one another.

The central concept of the paper is a multivalued group-like system called a polygroup. This special type of multigroup (in the sense of Dresher and Ore [8]) is defined in section 2 and examples are given to indicate how these systems occur naturally in various context. In section 3 polygroups are characterized as the atomic structures of complete atomic integral relation algebras. The Tarski complex-algebra construction [10] gives a full embedding (and even more) of polygroups into relation algebras. A variation of this result is true for cylindric algebras but will not be treated here. The significance of the embedding is that certain combinatorial and model-theoretic properties of polygroups automatically transfer to *RA*'s (and *CA*'s). This process can be used to provide a simple and unified treatment of certain model-theoretic/combinatorial results that have been obtained for both relation algebras and cylindric algebras. For example, Monk's work in [15] and [16] turns out to be just two interpretations of a simple polygroup result. A polygroup treatment of McKenzie's important work [14] has been carried out by the author in [7].

In addition to their close connection with integral relation algebras, polygroups are important for other reasons. Of particular interest is a class of systems derived from color schemes (see Example 3). Color schemes generalize the notion

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of (homogeneous) coherent configurations due to D. G. Higman [9] and the Bose–Mesner notion of an association scheme [2]. In section 4 properties of the polygroup derived from a color scheme $\mathcal{V}$ are related to properties of the automorphism group of $\mathcal{V}$. These ideas are used in section 5 to compute the polygroups associated with random color schemes forbidding a prescribed class of colored triangles. The last section comments on two combinatorial questions about permutation groups using the schemes introduced by Sims [20]. We determine the possible polygroups of the color schemes associated with certain rank 4 primitive groups (if they exist).

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2. Definition and examples of polygroups

The notion of a polygroup defined below is the same as the concept of a regular, reversible-in-itself multigroup with an absolute unit in the sense of Dresher and Ore [8]. However, "polygroup" is shorter. In [5] and [7] the systems were called pseudogroups by the author. In retrospect it appears the notion has played a fundamental role for many years. It appears implicitly in the work of Jónsson [11], Lyndon [12], and McKenzie [13] and [14].

DEFINITION. A *polygroup* is a system $\mathcal{M} = \langle M, \cdot, ^{-1}, e\rangle$ where $e \in M$, $^{-1}$ is a unary operation on M and $\cdot$ is a binary operation on M such that $x \cdot y$ is a non-empty subset of M for every pair $(x, y) \in M^2$ and the following axioms hold for all $x, y, z \in M$:

(P_1) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,
(P_2) $x \cdot e = x = e \cdot x$,
(P_3) $e \in x \cdot x^{-1}$ and $e \in x^{-1} \cdot x$,
(P_4) $x \in y \cdot z$ implies $y \in x \cdot z^{-1}$ and $z \in y^{-1} \cdot x$.

The statements above employ some obvious conventions:

(1) For $A, B \subseteq M$, $A \cdot B = \bigcup \{a \cdot b : a \in A, b \in B\}$ and
(2) elements of M are identified with singletons to prevent a proliferation of brackets.

As an example of the elementary consequences of the axioms we mention that $(x^{-1})^{-1} = x$ holds for all x in a polygroup. A polygroup in which every element has order 2 (i.e., $x^{-1}=x$ for all x) is called *symmetric*. As in group theory it can be shown that a symmetric polygroup is commutative.

EXAMPLE 1. *Double coset algebras.* Suppose G and H are groups with H a subgroup of G. Form a system $\langle M, \cdot, ^{-1}, H\rangle$ where $M=\{HgH : g\in G\}$, the collection of double cosets, $(HgH)^{-1}=Hg^{-1}H$, and

$$(Hg_1H)\cdot(Hg_2H)=\{Hg_1hg_2H : h\in H\}.$$

This system is a polygroup denoted by $G/\!/H$ (Dresher and Ore [8]).

EXAMPLE 2. *Prenowitz algebras.* Suppose $\mathscr{G}$ is a projective geometry with a set P of points and suppose, for $p\neq q$, $\overline{pq}$ denotes the set of all points on the unique line through p and q. Choose an object $I\notin P$ and form the system

$$P_{\mathscr{G}}=\langle P\cup\{I\}, \cdot, ^{-1}, I\rangle$$

where $x^{-1}=x$ and $I\cdot x=x\cdot I=x$ for all $x\in P\cup\{I\}$ and, for $p, q\in P$,

$$p\cdot q=\begin{cases}\overline{pq}-\{p,q\} & \text{if} \quad p\neq q\\ \{p, I\} & \text{if} \quad p=q\end{cases}$$

$P_{\mathscr{G}}$ is a polygroup (Prenowitz [18]). Important combinatorial properties of these systems were discovered by Lyndon [12] who phrased the ideas in terms of relation algebras. Other geometries give rise to polygroups in the same way, e.g., spherical geometry. In fact, any join space with identity (see Prenowitz and Jantosciak [19]) is a polygroup.

The following class of examples will be extremely important for us.

EXAMPLE 3. *Chromatic polygroups.* These system are derived from a very special type of edge-coloring of a complete graph. Suppose $\mathscr{C}$ is a non-empty set of colors and ε is a involution on $\mathscr{C}$. A *color scheme* is a system $\mathscr{V}=\langle V, C_a\rangle_{a\in\mathscr{C}}$ where each C_a is a binary relation on V and

(1) $\{C_a : a\in\mathscr{C}\}$ is a partition of $\{(x, y)\in V^2 : x\neq y\}$,
(2) $C_{\varepsilon(a)}=C_a^{\cup}$ for each $a\in\mathscr{C}$ ($^{\cup}$ is relation converse),
(3) each vertex has an edge of each color emanating from it (i.e., for every $a\in\mathscr{C}$, $x\in V$, $(x, y)\in C_a$ for some $y\in V$),
(4) If $a, b, c\in\mathscr{C}$ and $(x, y)\in C_c$, the existence of an (a, b)-path from x to y is independent of the choice of $(x, y)\in C_c$. That is, if $C_c\cap(C_a \mid C_b)\neq\varnothing$, then $C_c\subseteq C_a\mid C_b$ where $\mid$ denotes relation composition.

The relation C_a is thought of as the set of directed edges with "color a" in the complete directed graph with no loops on the set V. The purpose of the involution ε is to guarantee that the color assigned to an edge (y, x) depends only on the color assigned the reverse directed edge (x, y) and not on the particular (x, y). It is convenient to say that colors a and $\varepsilon(a)$ are *paired*. In the special case where colors are *self-paired* (i.e., $\varepsilon(a)=a$ for all a), the colors schemes can be conveniently pictured by coloring the edges of undirected graphs. A system $\mathcal{V}=\langle V, C_a\rangle_{a\in\mathcal{C}}$ that satisfies (1) and (2) is called a *partial color scheme.*

Before introducing the polygroup we mention two widely studied special cases of the notion of color scheme.

A. Homogeneous coherent configurations (D. G. Higman [9]) are obtained by strengthening (4) as follows:

(4′) for $a, b, c \in \mathcal{C}$ and $(x, y) \in C_c$ the *number* of (a, b)-paths from x to y is independent of the choice of $(x, y) \in C_c$.

The numbers obtained in (4′) are called *intersection numbers.* They allow these systems to be studied via matrix algebra.

B. Association schemes (Bose and Mesner [2]) are homogeneous coherent configurations with $\varepsilon(a)=a$ for all $a \in \mathcal{C}$. Among the important association schemes are those associated with distance-transitive and strongly regular graphs (Biggs [1] and Cameron and Van Lint [3]).

Now, suppose $\mathcal{V}=\langle V, C_a\rangle_{a\in\mathcal{C}}$ is a color scheme and choose a new symbol $I\notin\mathcal{C}$. (It is safe to think of I as the identity relation on V.) The *color algebra* of $\mathcal{V}$ is the system

$$\mathcal{M}_{\mathcal{V}}=\langle\mathcal{C}\cup\{I\}, \cdot, {}^{-1}, I\rangle$$

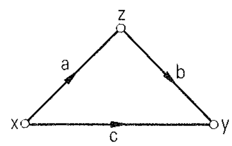
where the inverse is defined by $a^{-1}=\varepsilon(a)$ for $a\in\mathcal{C}$ and $I^{-1}=I$ and the product is defined by $x\cdot I=I\cdot x=x$ for $x\in\mathcal{C}\cup\{I\}$ and

$$a\cdot b=\{c\in\mathcal{C}: C_c\subseteq C_a \mid C_b\}\cup\{I: b=a^{-1}\}$$

for $a, b\in\mathcal{C}$.

It is straightforward to verify $\mathcal{M}_{\mathcal{V}}$ is a polygroup. A polygroup is called *chromatic* if it is isomorphic to a system $\mathcal{M}_{\mathcal{V}}$ derived from some color scheme $\mathcal{V}$.

The following terminology is convenient. An element of $\mathcal{C}^3$ is a *(colored) triangle.* An *edge* of $\mathcal{V}$ is a pair $(x, y)\in V^2$ with $x\neq y$. A triangle $(a, b, c)\in\mathcal{C}^3$ is *realizable on an edge* (x, y) of $\mathcal{V}$ if $(x, y) \in C_c \cap (C_a \mid C_b)$, i.e., for some z we have the picture



A class K of colored triangles is *locally realizable* in $\mathcal{V}$ if for every edge (x, y) of $\mathcal{V}$ and every triangle $(a, b, c) \in K$ with $(x, y) \in C_c$, the triangle is realizable on (x, y) in $\mathcal{V}$. A class K of colored triangles is *forbidden* in $\mathcal{V}$ (or $\mathcal{V}$ *forbids* K) if no triangle in K is realizable on any edge in $\mathcal{V}$.

Remarks. 1. Condition (4) in the definition of color scheme means that if a triangle is not forbidden, it is locally realizable.

2. Given a polygroup $\mathcal{M}$, $M' = M \setminus \{e\}$ can be regarded as a set of colors. Then $\mathcal{M}$ is completely determined by a class of colored triangles

$$FC(\mathcal{M}) = \{(a, b, c) \in (M')^3 : c \notin a \cdot b\}$$

called the *forbidden class* of $\mathcal{M}$. Note that $\mathcal{M}$ is chromatic iff there exist a color scheme $\mathcal{V}$ that forbids $FC(\mathcal{M})$ and locally realizes its complement $FC(\mathcal{M})^c$.

3. In general, if a triangle (a, b, c) is realized in a color scheme, so are the following ones: $(\varepsilon(a), c, b), (b, \varepsilon(c), \varepsilon(a)), (\varepsilon(b), \varepsilon(a), \varepsilon(c)), (c, \varepsilon(b), a), (\varepsilon(c), a, \varepsilon(b))$. When we speak of a class of colored triangles, we assume it is closed under the above "symmetries." In case the colors are self-paired this just means that when (a, b, c) belongs to a class so does all rearrangements of (a, b, c). When writing down a class of colored triangles it is convenient to list only one triangle from among the equivalent symmetric ones.

Example 1 is now generalized.

EXAMPLE 4. Double quotients of polygroups. At times we want to construct new polygroups as quotients of old ones.

DEFINITION. Let θ be an equivalence relation on a polygroup $\mathcal{M}$. Then

(1) θ is a full conjugation on $\mathcal{M}$ if
  (i) $x\theta y$ implies $x^{-1}\theta y^{-1}$;
  (ii) $z \varepsilon x \cdot y$ and $z\theta z'$ implies there exist $x'\theta x$, $y'\theta y$ with $z' \varepsilon x' \cdot y'$.

(2) θ is called a special conjugation if it satisfies (i), (ii) and, in addition,
  (iii) $x\theta e$ implies $x = e$.

The following shows that full conjugation is the proper notion.

PROPOSITION 2.1 (Theorem 8.6 in [7]). *For a equivalence relation θ on a polygroup $\mathcal{M}$, θ is a full conjugation on $\mathcal{M}$ iff the system $\langle\{\theta x : x \in M\}, \cdot, ^{-1}, \theta e\rangle$, where $\cdot$ and $^{-1}$ are the induced operations on θ-classes, is a polygroup.*

The system in Proposition 2.1 is denoted $\mathcal{M} /\!/ \theta$ and called a *(double) quotient of* $\mathcal{M}$. Double quotients of groups are particularly important. Let $Q^2(\text{Group})$ consist of all polygroups isomorphic to a double quotient of a group and let $Q_s^2(\text{Group})$ consist of all polygroups isomorphic to $G /\!/ \theta$ where θ is a special conjugation of a group G.

Some of the ways to obtain full conjugations on a group G are indicated below:

(a) A congruence relation θ on G is a full conjugation. $G /\!/ \theta$ is just the usual quotient group in this case.

(b) If H is a subgroup of G and we define

$$x\theta_H y \text{ iff } x \text{ and } y \text{ generate the same double coset (i.e., } HxH = HyH),$$

then θ_H is a full conjugation on G and $G /\!/ \theta_H = G /\!/ H$.

(c) Define $x\theta y$ iff x and y are conjugate in the usual sense with respect to a subgroup H (i.e., there exist $h \in H$, $y = h^{-1}xh$). Then θ is a special conjugation. This example generalizes as follows.

(d) Suppose K is a group of automorphisms of G. Define $x\theta y$ iff $y = x\sigma$ for some $\sigma \in K$. This is also a special conjugation on G.

Special conjugations of groups were used by Utumi [21] to obtain important examples of cogroups.

Analogues of the standard homomorphism and isomorphism theorems are obtained for double quotients in [7]. As a consequence of these the connection between double quotients and double cosets can be easily seen.

PROPOSITION 2.2. *If θ is a full conjugation on a group G, then $\theta e = H$ is a subgroup of G and $G /\!/ \theta \cong (G /\!/ H) /\!/ \psi$ for some special conjugation ψ on $G /\!/ H$.*

Double quotients of groups are related to chromatic polygroups.

THEOREM 2.3. *Every polygroup in $Q^2(\text{Group})$ is chromatic.*

Proof. Suppose θ is a full conjugation on a group G. By 2.2 $\theta e = H$ is a subgroup of G. Let $\mathscr{C}=\{\theta g:\theta g\neq H\}$, $\varepsilon(a)=a^{-1}$ for $a\in\mathscr{C}$, $V=\{Hx:x\in G\}$, and for each $a\in\mathscr{C}$ set

$$C_a=\{(Hx, Hy)\in V^2: xy^{-1}\in a\}.$$

It is easily seen that $\mathscr{V}=\langle V, C_a\rangle_{a\in\mathscr{C}}$ is a color scheme. If $c\in a\cdot b$ (in $G/\!/\theta$) and $(Hu, Hv)\in C_c$, there exist $r\in a$ and $s\in b$ with $uv^{-1}=rs$. Letting $z=r^{-1}u$ it follows that $(Hu, Hz)\in C_a$ and $(Hz, Hv)\in C_b$ so $(Hu, Hv)\in C_a\mid C_b$. Conversely, if $C_c\subseteq C_a\mid C_b$ and $x\in c$, then $(Hx, H)\in C_c$ so there exists a $z\in b$ with $xz^{-1}\in a$. Hence $x\in a\cdot b$ and so $c\subseteq a\cdot b$ (in $G/\!/\theta$). It now easily follows that the natural map from $G/\!/\theta$ onto $\mathscr{M}_\mathscr{V}$ that sends θg to θg and θe to I (=identity of $\mathscr{M}_\mathscr{V}$) is an isomorphism.

The color scheme used in 2.3 will be called the *regular color scheme* representation of $G/\!/\theta$. In the case of a double coset algebra $G/\!/H$ Theorem 2.3 can be strengthened to represent $G/\!/H$ as $\mathscr{M}_\mathscr{V}$ where $\mathscr{V}$ is a homogeneous coherent configuration (D. G. Higman [9]).

3. Connections with relation algebras

The notion of a relation algebra is due to Tarski. We use the following definitions from [10], chapter 4. A *relation algebra* (*RA*) is a system $\langle A, +, \cdot, -, 0, 1, ;, ^{\cup}, 1'\rangle$ where $\langle A, +, \cdot, -, 0, 1\rangle$ is a Boolean algebra and the following axioms hold:

(R_1) $(x;y);z=x;(y;z)$,
(R_2) $1';x=x;1'=x$,
(R_3) $(x;y)\cdot z=0$ iff $(x^{\cup};z)\cdot y=0$ iff $(z;y^{\cup})\cdot x=0$.

The main examples of relation algebras are systems $\langle\mathscr{A},\cup,\cap,\sim,\varnothing, X^2, |, ^{\cup}, I_X\rangle$ where $\mathscr{A}$ is a collection of binary relations on the set X that contains $\varnothing$, X^2 and $I_X=\{(x,x):x\in X\}$ and is closed under $\cup$, $\cap$, $\sim$, relation composition $|$, and converse $^{\cup}$. Such a system is called a *proper relation algebra*. A relation algebra is *representable* if it is isomorphic to a subdirect product of proper relation algebras. A relation algebra with the property

$$x;y=0 \quad\text{implies}\quad x=0 \quad\text{or}\quad y=0$$

is called an integral *RA* (abbreviated *IRA*). The integral condition is equivalent to: the element 1' is an atom. It implies the algebra is simple (see [10]), thus an *IRA* is representable if it is isomorphic with a proper relation algebra.

The *complex algebra* of a polygroup $\mathcal{M}=\langle M, \cdot, ^{-1}, e\rangle$ is the system $\mathfrak{A}[\mathcal{M}]=\langle \mathcal{P}(M), \cup, \cap, \sim, \emptyset, M, \cdot, ^{-1}, \{e\}\rangle$ where $\langle \mathcal{P}(M), \cup, \cap, \sim, \emptyset, M\rangle$ is the Boolean algebra of all subsets of M and $\cdot$ and $^{-1}$ denote the extensions of the polygroup operations to subsets.

The complex algebra construction gives a one–one correspondence (up to isomorphism) between polygroups and complete atomic *IRA*'s.

THEOREM 3.1. (1) *$\mathfrak{A}[\mathcal{M}]$ is a complete atomic IRA for every polygroup $\mathcal{M}$.*

(2) *For every complete atomic IRA $\mathfrak{A}$ the system $At(\mathfrak{A})=\langle At_{\mathfrak{A}}, ;, ^{\cup}, 1'\rangle$, where $At_{\mathfrak{A}}$ is the set of atoms of $\mathfrak{A}$, is a polygroup.*

(3) *If $\mathcal{M}$ is a polygroup and $\mathfrak{A}$ is a complete atomic IRA, then*

$$\mathcal{M}\cong At(\mathfrak{A}[\mathcal{M}]) \quad \textit{and} \quad \mathfrak{A}\cong \mathfrak{A}[At(\mathfrak{A})].$$

Proof. (1) Only (R_3) needs justification. If $(X\cdot Y)\cap Z\neq\emptyset$, then there exist $a\in Z$, $b\in X$, $c\in Y$ with $a\in b\cdot c$. By (P_4), $b\in a\cdot c^{-1}$ so $(Z\cdot Y^{\cup})\cap X\neq\emptyset$. The other implications are similar using (P_4) and $(x^{-1})^{-1}=x$.

(2) Observe that $a;b$ and $a^{\cup}$ are atoms whenever a, b are atoms. (P_1), (P_2) and (P_4) follow from (R_1), (R_2), and (R_3) respectively. For (P_3) observe that if $a\in At_{\mathfrak{A}}$, $(1';a)\cdot a\neq 0$ so $(a;a^{\cup})\cdot 1'\neq 0$ by (R_3) and hence $1'\in a;a$ since $1'$ is an atom.

(3) The correspondence of x with $\{x\}$ gives the first isomorphism and the map that sends an element $a\in A$ to the set of atoms $x\leq a$ gives the second.

Since every *IRA* is embeddable in a complete atomic *IRA* (see [10]) we obtain

COROLLARY 3.2. *The class of IRA's coincides with the class of all subalgebras of complex algebras of polygroups.*

The class of polygroups forms a category where a morphism from $\mathcal{M}$ to $\mathcal{N}$ means that $\mathcal{N}$ is isomorphic to a *special* double quotient of $\mathcal{M}$. The "homomorphism" notion corresponding to the double quotient construction is worked out in [7] but it is not essential here. The class of complete atomic *IRA*'s can form a category in several ways. When we refer to this class as a category we intend that the morphisms are complete embeddings, i.e., embeddings that are completely additive.

THEOREM 3.3. *The complex algebra construction is a functor that gives a dual equivalence between the category of polygroups and the category of complete atomic IRA's.*

Proof. It is enough to consider how the morphisms work. If θ is a special conjugation on a polygroup $\mathcal{M}$, then $\mathfrak{A}[\mathcal{M}/\!/\theta]$ is completely embeddable in $\mathfrak{A}[\mathcal{M}]$. This is clear since the atoms of $\mathfrak{A}[\mathcal{M}/\!/\theta]$ (as elements of $\mathcal{M}/\!/\theta$) give rise to a partition of M such that $\{e\}$ is a θ-class (recall θ is special). On the other hand, suppose f is a complete embedding of $\mathfrak{A}[\mathcal{N}]$ into $\mathfrak{A}[\mathcal{M}]$. For $a \in M$ let $h(a)$ be the unique element $b \in N$ such that $a \in f(\{b\})$. Clearly h maps M onto N and $h(e) = e$. Moreover, h is a special homomorphism in the sense of [7] and the Fundamental Homomorphism Theorem gives $\mathcal{N} \cong \mathcal{M}/\!/\ker h$ where $\ker h$ is a special conjugation on $\mathcal{M}$. Alternately, one can argue directly that $\mathcal{N} \cong \mathcal{M}/\!/\theta$ where θ is defined on M by: $x\theta y$ iff $h(x) = h(y)$.

The categorical duality in 3.2 gives a way to look at special classes of *IRA*'s and polygroups.

COROLLARY 3.4. (1) *A polygroup $\mathcal{M}$ is in $Q_s^2(Group)$ iff $\mathfrak{A}[\mathcal{M}]$ is group representable* (*see* [10]).

(2) *A polygroup $\mathcal{M} \in Q^2(Group)$ iff $\mathfrak{A}[\mathcal{M}]$ is permutational* (*see McKenzie* [14]).

(3) *A polygroup $\mathcal{M}$ is chromatic iff $\mathfrak{A}[\mathcal{M}]$ has a completely additive representation.*

Statements (1) and (2) are clear using 2.2. The most interesting is (3). If $\mathcal{V} = \langle V, C_a \rangle_{a \in \mathcal{C}}$ is a color scheme, a completely additive representation on $\mathcal{V}$ for $\mathfrak{A}[\mathcal{M}_{\mathcal{V}}]$ can be obtained by assigning each color $a \in C$ the relation C_a (and assigning I to I_V) and extending by additivity. Conversely, $\mathfrak{A}[\mathcal{M}]$ is simple so any completely additive representation corresponds to a partition $\{R_m : m \in M\}$ of X^2 for some set X. $\mathcal{V} = \langle X, R_m \rangle_{m \in M'}$ is a color scheme and the *RA* representation restricts to give $\mathcal{M} \cong \mathcal{M}_{\mathcal{V}}$.

Remarks. 1. The notion of polygroup can be extended by replacing e with a "set of identity elements." The resulting systems can be used to characterize all relation algebras in the same way we have described integral *RA*'s using polygroups in this section. This has independently been worked out by Brian McEvoy and the author. Inhomogeneous configurations of D. G. Higman [9] give rise to interesting models of this more general concept.

2. Polygroups can also be generalized by dropping the associative law. In this case the notion of multivalued loop results. These systems are investigated in [7] and can be used to characterize integral CA_3's with the help of a "cylindric" complex algebra construction and an "adjunction construction" (announced in [4]).

4. Automorphism groups of color schemes

The important classes of chromatic polygroups we have met are characterized in terms of the automorphism groups of color schemes. An automorphism of a color schemes $\mathcal{V}=\langle V, C_a\rangle_{a\in\mathscr{C}}$ is a permutation σ of V such that, for all $a\in\mathscr{C}$, $x, y\in V$, $(x, y)\in C_a$ iff $(x\sigma, y\sigma)\in C_a$.

THEOREM 4.1. *For a polygroup $\mathcal{M}$, $\mathcal{M}\in Q^2(Group)$ iff $\mathcal{M}\cong\mathcal{M}_{\mathcal{V}}$ for some color scheme $\mathcal{V}$ with* Aut$(\mathcal{V})$ *transitive on vertices.*

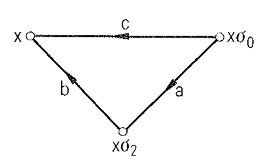
Proof. First note that if $\mathcal{V}$ is the regular color scheme representing $G/\!/\theta$ (recall 2.3), then G acts transitively on the coset space V by right multiplication. Now suppose $\mathcal{M}$ is represented by a color scheme $\mathcal{V}=\langle V, C_a\rangle_{a\in M'}$ with Aut$(\mathcal{V})=G$ transitive on V. Fix $x\in V$ and partition $V=\bigcup\{V_a : a\in M\}$ where $V_e=\{x\}$ and $V_a=\{y\in V:(x, y)\in C_a\}$ for $a\neq e$. Define an equivalence relation $\theta^{\mathcal{V}}$ on G by:

$$\sigma\theta^{\mathcal{V}}\tau \quad \text{iff,} \quad \text{for every } a\in M,\ x\sigma\in V_a \text{ iff } x\tau\in V_a.$$

The $\theta^{\mathcal{V}}$-classes correspond one–one with elements of $\mathcal{M}$. Namely, $a\in M$ corresponds to the $\theta^{\mathcal{V}}$-class $G_{xa}=\{\sigma\in G: x\sigma^{-1}\in V_a\}$ where, of course, $G_{xe}=G_x$ the stabilizer of G at x. The elements of V correspond in a natural way to cosets of G_x. Namely, for $y\in V$, $G_{xy}=\{\sigma\in G: y=x\sigma\}=G_x\tau$ where τ is any element of G_{xy}.

LEMMA. *$\theta^{\mathcal{V}}$ is a full conjugation on G.*

It is easily checked that $\theta^{\mathcal{V}}$ preserves inverses. We suppose $\sigma_0=\sigma_1\sigma_2$ and $\sigma_0'\theta^{\mathcal{V}}\sigma_0$ and show $\sigma_0'=\sigma_1'\sigma_2'$ for some $\sigma_1'\theta^{\mathcal{V}}\sigma_1$ and $\sigma_2'\theta^{\mathcal{V}}\sigma_2$. Suppose $\sigma_0\in G_{xc}$, $\sigma_1\in G_{xa}$ and $\sigma_2\in G_{xb}$. We have



in $\mathcal{V}$. For, if $(x\sigma_0, x\sigma_2)\in C_d$ for some d, applying σ_2^{-1} yields $(x\sigma_1, x)\in C_d$ (since $\sigma_0=\sigma_1\sigma_2$); but $(x\sigma_1, x)\in C_a$ so $d=a$. Consequently, the colored triangle (a, b, c) is locally realizable in $\mathcal{V}$.

Now, $\sigma_0'\theta^{\mathcal{V}}\sigma_0$. Let $y=x\sigma_0'$. Since (a, b, c) is locally realizable and $(y, x)\in C_c$, there exist z such that $(y, z)\in C_a$ and $(z, x)\in C_b$. G vertex-transitive implies $z=x\tau$ for some $\tau\in G_{xb}$. Choose $\mu\in G$ so that $x\mu=y\tau^{-1}$ (G vertex-transitive). Now

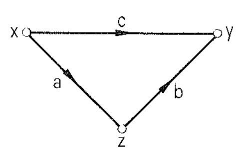
$(y, x\tau) \in C_a$ implies $(x\mu, x) \in C_a$ so $\mu \in G_{xa}$. Since $x\mu\tau = y$, both $\mu\tau$ and σ_0' belong to the coset G_{xy}. Thus, $\sigma_0' = \nu\mu\tau$ for some $\nu \in G_x$ from which it follows that $\sigma_0' = \sigma_1'\sigma_2'$ where $\sigma_1' = \nu\mu\theta^{\mathcal{V}}\sigma_1$ and $\sigma_2' = \tau\theta^{\mathcal{V}}\sigma_2$ as required.

To finish the theorem we must show the natural bijection that sends $a \in \mathcal{M}$ to $G_{xa} \in G /\!/ \theta^{\mathcal{V}}$ is an isomorphism. First, we check that $e \in a \cdot b$ iff $G_x \subseteq G_{xa}G_{xb}$ (so that inverses correspond). If $e \in a \cdot b$ (so $a = b^{-1}$) and $\sigma \in G_{xa}$, then $(x, x\sigma) \in C_{a^{-1}} = C_b$. So $1_V = \sigma\sigma^{-1} \in G_{xa}G_{xb}$ and therefore $G_x \subseteq G_{xa}G_{xb}$. Conversely, if $G_x \subseteq G_{xa}G_{xb}$, $1_V = \sigma\tau$ for some $\sigma \in G_{xa}$, $\tau \in G_{xb}$. Since $(x\sigma, x) \in C_a$, applying τ yields $(x, x\tau) \in C_a$. Therefore, $(x\tau, x) \in C_{a^{-1}} \cap C_b$ so $b = a^{-1}$.

It remains to show

$$c \in a \cdot b \quad \text{iff} \quad G_{xc} \subseteq G_{xa}G_{xb}$$

where we may assume $a, b, c \neq e$. The argument from left to right is essentially the same as used in the last step of the lemma above. We suppose $G_{xc} \subseteq G_{xa}G_{xb}$. By the product definition in $\mathcal{M}_{\mathcal{V}}$ it suffices to realize an (a, b, c) triangle on some $(x, y) \in C_c$. Choose σ such that $x = y\sigma$ (by transitivity of G). Then $\sigma \in G_{xc} \subseteq G_{xa}G_{xb}$ so $\sigma = \tau\mu$ for some $\tau \in G_{xa}$, $\mu \in G_{xb}$. Let $z = x\tau^{-1}$. Then we have



in $\mathcal{V}$. Clearly $(x, z) \in C_a$. Suppose $(z, y) = (x\tau^{-1}, x\mu^{-1}\tau^{-1}) \in C_d$ for some d. Applying τ we obtain $(x, x\mu^{-1}) \in C_d$ from which $d = b$ follows. Thus, $c \in a \cdot b$ as desired.

A group G of automorphisms of a graph is called *strongly transitive on edges* if, for every pair of edges (x, y) and (u, v) there exist $\sigma \in G$ such that $x\sigma = u$ and $y\sigma = v$. This is equivalent to G being vertex-transitive and the stabilizer at a vertex, x being transitive on the vertices adjacent to x. For directed graphs the notion is the same as edge-transitive but for undirected graphs the notion is stronger. The next result is a polygroup version of Lemmas 1.1 and 1.2 in McKenzie [14].

THEOREM 4.2. (1) *$\mathcal{M}$ is isomorphic to a double coset algebra Iff $\mathcal{M} \cong \mathcal{M}_{\mathcal{V}}$ for some color scheme $\mathcal{V}$ where* Aut $(\mathcal{V})$ *is strongly transitive on the edges of each monochrome subgraph.*

(2) *$\mathcal{M} \in Q_s^2(Group)$ iff $\mathcal{M} \cong \mathcal{M}_{\mathcal{V}}$ for some color scheme $\mathcal{V}$ for which there exists $G \subseteq$* Aut $(\mathcal{V})$ *such that* (i) *G is vertex-transitive and* (ii) *$G_x = \{1_V\}$ for all $x \in V$.*

Proof. (1) Consider the regular color scheme representing $G /\!/ H$. If $(Hx, H), (Hy, H) \in C_a$, both x and y belong to the double coset a; so Hy can be obtained from Hx by right multiplication by an element of H. Thus, the regular color scheme has the desired properties. Conversely, if $\mathrm{Aut}(\mathscr{V})$ is vertex-transitive, $\mathscr{M}_{\mathscr{V}} \cong G /\!/ \theta$ (Theorem 4.1) where $G = \mathrm{Aut}(\mathscr{V})$ and each θ-block G_{xa} is a union of cosets $G_x\tau$ (where $x\tau^{-1} \in V_a$). G_{xa} is a single double coset since G_x is transitive on V_a.

(2) If $\mathscr{M} = G /\!/ \theta$ where $\theta e = \{e\}$, the vertex set in the regular color scheme representing $\mathscr{M}$ is identifiable with G and the action of G on G is regular (i.e., $G_x = \{1\}$ for all x). Conversely, suppose $\mathscr{V}$ representing $\mathscr{M}$ has (i) and (ii). Note that the definition of $\theta^{\mathscr{V}}$ in 4.1 and the proof that it is a full conjugation depends only on G being a transitive subgroup of $\mathrm{Aut}(\mathscr{V})$. Thus (i) implies $\mathscr{M} \cong G /\!/ \theta$ and (ii) implies θ is special.

5. Random color schemes with forbidden triangles

In section 3 relation algebras were related to polygroups, representable *RA*'s with chromatic polygroups, etc. In this and the following section we give some examples to show that polygroups can produce new results, new insights, not only for relation algebras but in other areas as well.

In [7] all polygroups with at most 4 elements are determined. The author has now examined the 102 isomorphism types of 4 element systems and, in all except 6 cases, has determined whether or not the system is chromatic. Several general techniques have evolved from this study. As a sample, one method, which generalizes the idea of a random graph, is presented here.

For convenience assume the set of colors involved is $\mathscr{C} = \{1, \ldots, n\}$ and the given involution ε carries i to i'. Suppose $\mathscr{V} = \langle V, C_i \rangle_{i \in \mathscr{C}}$ is a color scheme and $K \subseteq C^3$ is a class of colored triangles. A sequence $F_1, \ldots, F_n$ of subsets of V is *compatible with* K if it is not the case there exist $i, j \in \mathscr{C}$, $u \in F_i$, $v \in F_j$ with $(u, v) \in C_k$ where $(i, j', k) \in K$. (Of course, $i = j$ is allowed.)

A color scheme $\mathscr{V}$ is called a *random n-color scheme with forbidden K* if

(i) no triangle in K is realized in $\mathscr{V}$,
(ii) for every sequence $F_1, \ldots, F_n$ of n finite pairwise disjoint sets of vertices compatible with K, there exists a vertex p such that for every $i \in \mathscr{C}$ and $v \in F_i$ $(v, p) \in C_i$ (and of course $(p, v) \in C_{i'}$).

A standard Cantor back-and-forth argument yields.

THEOREM 5.1. *For a class K of colored triangles, any two denumerable random n-color schemes with forbidden K are isomorphic. Moreover, the automorphism group of such a color scheme is strongly transitive on the edges of each monochrome subgraph.*

For 4.2 we obtain

COROLLARY 5.2. *The color algebra of a random color scheme with forbidden K is a double coset algebra.*

The following provides an easy sufficient condition for a polygroup to be represented by a random color scheme.

THEOREM 5.3. *A random n-color scheme with forbidden K exists if K depends on at most $n-1$ colors (i.e., one of the n colors does not occur in any triangle of K).*

Proof. Suppose the triangles in K involve only the colors $1, \ldots, n-1$. (I.e., color n is not used. Using the symmetries, n' is also not used; of course, n' could equal n depending on the involution.) Start the construction with a trivial partial color scheme, say

$$\langle V_0, C_1^0, \ldots, C_n^0\rangle$$

where

$$V_0 = \{v_0, v_1\}, \qquad (v_0, v_1) \in C_n^0$$

and

$$(v_1, v_0) \in C_{n'}^0.$$

The other C_i^0's are empty. By induction suppose we have a finite partial color scheme $\langle V_k, C_1^k, \ldots, C_n^k\rangle = \mathcal{V}_k$ with the property that no triangles in K are realized in $\mathcal{V}_k$. Let

$$\mathcal{F} = \{(F_1, \ldots, F_n) \in \mathcal{P}(V_k)^n : F_1, \ldots, F_n \text{ are compatible with } K \text{ and } \{F_1, \ldots, F_n\} \text{ partition } V_k\}.$$

For each $F = (F_1, \ldots, F_n) \in \mathcal{F}$ introduce a new point v_F and set

$$V_{k+1} = V_k \cup \{v_F : F \in \mathcal{F}\}.$$

Order the elements of $\mathscr{F}=\{F^1, \ldots, F^m\}$. Now, for $i<n$, $i \neq n'$, set

$$C_i^{k+1}=C_i^k \cup\{(p, v_F): F \in \mathscr{F} \text{ and } p \in F_i\} \cup\{(v_F, p): F \in \mathscr{F}, p \in F_{i'}\}$$
$$C_n^{k+1}=C_n^k \cup\{(p, v_F): F \in \mathscr{F}, p \in F_n\} \cup\{(v_F, p): F \in \mathscr{F}, p \in F_{n'}\} \cup\{(v_{F^r}, v_{F^s}): r<s\}$$
$$C_{n'}^{k+1}=(C_n^{k+1})^{\cup}.$$

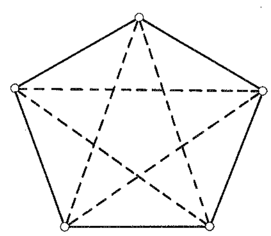
No triangles in K can be realized in

$$\mathscr{V}_{k+1}=\langle V_{k+1}, C_1^{k+1}, \ldots, C_n^{k+1}\rangle$$

so we may continue the induction. It follows that $V=\bigcup_{j<\omega} V_j$ is denumerable and the system $\langle V, C_1, \ldots, C_n\rangle$, where $C_i=\bigcup_{j<\omega} C_i^j$ (for $i=1, \ldots, n$), is a random n-color scheme with forbidden K.

Remarks. 1. The notion of a random color scheme forbidding $\varnothing$ agrees with the usual idea of a random coloring (a random graph in case $n=2$).

2. The condition in 5.3 is not necessary (pointed out by P. J. Cameron.) For example, the 2-color scheme for the pentagon



is random with forbidden class $K=\{(1,1,1),(2,2,2)\}$. There are other examples as well.

6. Connections with finite permutation groups

In view of 3.4 the results in section 5 have obvious applications to the study of relation algebras. In this last section we want to indicate how polygroups can also provide information about finite permutation groups.

For a brief introduction to the relation between permutation groups and edge-colored graphs the survey by P. M. Neumann [17] is recommended. We recall a few basic ideas. Suppose G is a transitive group of permutations on a set V and that $\Delta_0, \ldots, \Delta_{r-1}$ are the orbits of G acting pointwise on $V \times V$ where

$\Delta_0=\{(x,x):x\in V\}$. For an $x\in V$, the relationship $\Gamma_i=\{y:(x,y)\in\Delta_i\}$ gives a one–one correspondence between the orbits $\Delta_0,\ldots,\Delta_{r-1}$ and orbits $\Gamma_0,\ldots,\Gamma_{r-1}$ of the stabilizer G_x acting on V. The Γ_i's are called the *suborbits* of G and the numbers $n_i=|\Gamma_i|$ are called the *subdegrees* of G. $\Gamma_0=\{x\}$ is the *trivial suborbit* and $n_0=1$. The number of suborbits (= number of orbits of G acting on $V\times V$) is called the *rank* of G.

The G-orbits $\Delta_1,\ldots,\Delta_{r-1}$ give a partition of $V^2-\Delta_0$. We regard these sets as a collection of colorings for the edges of the complete graph on V. $\langle V,\Delta_1,\ldots,\Delta_{r-1}\rangle$ is a color scheme in the sense of Example 3. Since the converse $\Delta_i^{\cup}$ of a G-orbit is again an orbit, the involution ε on the colors $\{1,\ldots,r-1\}$ is the natural map $\Delta_{\varepsilon(i)}=\Delta_i^{\cup}$. The colors correspond one–one with the non-trivial suborbits and suborbits Γ_i and $\Gamma_{\varepsilon(i)}$ are said to be *paired* in agreement with the terminology for color schemes.

The above color schemes were introduced by Sims [20]. In particular, the automorphism group of the scheme is strongly transitive on the edges of each monochrome subgraph (Sims [20], also see Wielandt [23] or Neumann [17]). It follows (Theorem 4.2) that the color algebra $\mathcal{M}_{\mathcal{V}}$ of such a scheme is isomorphic to the double algebra $G/\!\!/G_x$.

For the applications in mind, G will be a primitive permutation group. A result of D. G. Higman [9] relates the primitivity of G to a property of the associated color scheme. Namely, a (finite) permutation group G is primitive on V iff every monochrome subgroup is connected. As a consequence of Higman's result if V is the Sims color scheme associated with the action of G we can tell from the multiplication table of the polygroup $\mathcal{M}_{\mathcal{V}}$ whether or not G is primitive.

We now wish to comment on a question about primitive groups raised by P. J. Cameron (Problem 1 in [17]). Does there exist a (finite) primitive group of rank 4 with two of its non-trivial subdegrees co-prime?

We show

THEOREM 6.1. *The Sims color scheme $\mathcal{V}$ of a primitive group G of rank 4 with two non-trivial co-prime subdegrees has color algebra $\mathcal{M}_{\mathcal{V}}$ isomorphic to either A_{12}^{02}, A_{13}^{02}, or A_4^{012}. (See Table.) Each of these polygroups are chromatic being represented by a (denumerable) random 3-color scheme with a forbidden class.*

Sketch of Proof. Rank 4 permutation groups give rise to 4 element polygroups, all isomorphism types of which have been described (Comer [7]). Higman's criteria allows us to decide when $\mathcal{M}_{\mathcal{V}}$ comes from a primitive group. A result of Marie J. Weiss [22] (see [17], Theorem 3) implies that if n_i, n_j are co-prime subdegrees of G with $n_i>n_j$, then $i\cdot j$ is a single value in $\mathcal{M}_{\mathcal{V}}$. Again, from the multiplication table it is trivial to tell if a polygroup has such elements. The three algebras above are the only chromatic ones that satisfy both Higman's and Weiss' conditions. The last statement follows from 5.2 and 5.3 by observing that 2 does not belong to the forbidden class of each algebra: $FC(A_{12}^{02}) = \{(1,1,1),(1,1,3),(1,3,3),(3,3,3)\}$, $FC(A_{13}^{02}) = \{(1,1,1),(1,1,3),(1,3,3)\}$ and $FC(A_4^{012}) = \{(1,1,3),(1,3,3)\}$.

Remarks. 1. The notation for the algebras comes from the enumeration in [7].

2. P. J. Cameron has shown (private communication) that no *finite* primitive group can have A_{12}^{02} as its color algebra. Thus, Theorem 6.1 can be improved to read: the Sims color scheme of a finite primitive group that satisfies the hypothesis of 6.1 (if one exists) has either A_{13}^{02} or A_4^{012} as its color algebra.

The second question we wish to comment on is whether or not there exists a primitive rank 4 permutation group with a non-self-paired orbit other than the projective special unitary group PSU(3, 3). PSU(3, 3) acts on a set with 36 points with orbit sizes 1, 7, 7, 21. We have

THEOREM 6.2. *The color scheme associated with a primitive rank 4 permutation group with a non-self-paired orbit has one of the following polygroup types:* C_6^{03} (*= type of* $PSU(3,3)$), C_4^{03}, C_{11}^{0123}, C_{13}^{0123}, C_{14}^{0123}, C_5^{012} *and possibly* C_8^{0123} *or* C_{10}^{0123}. (*See Tables.*)

Sketch of Proof. Higman's criteria and the non-self-paired requirement yield the above polygroups from the classification in [7].

Remarks. 1. The forbidden class of the first 5 algebras listed in 6.2 does not involve all 3 colors. From 5.3 and 5.2 it follows that these algebras are double coset polygroups represented by a denumerable random 3-color scheme forbidding the appropriate class. C_5^{012} can be shown to be chromatic but it is not known whether or not it is a double coset algebra. It is not known whether or not C_8^{0123} and C_{10}^{0123} are chromatic. (Hence it is possible they can be deleted from the list in 6.2).

2. From the existence of the random color schemes above we know there are denumerable primitive rank 4 groups with a non-self-paired orbit. It would be valuable to know, for a random n-color scheme $\mathscr{V}$ with forbidden class K, when there exist a finite scheme $\mathscr{W}$ with $\mathscr{M}_{\mathscr{V}} \cong \mathscr{M}_{\mathscr{W}}$.

3. It is hoped the results in this section can aid group theorists to solve the problems mentioned. While the polygroups do not disclose the exact intersection numbers in a coherent configuration, they do tell which are zero and which are non-zero.

Tables. (The identity rows and columns are omitted.)

A^{02}_{12}	1	2	3
1	02	123	2
2	123	0123	123
3	2	123	02

A^{02}_{13}	1	2	3
1	02	123	2
2	123	0123	123
3	2	123	023

A^{012}_{4}	1	2	3
1	012	123	2
2	123	0123	123
3	2	123	023

C^{03}_{4}	1	2	3
1	3	03	123
2	03	3	123
3	123	123	0123

C^{03}_{6}	1	2	3
1	23	03	123
2	03	13	123
3	123	123	0123

C^{012}_{5}	1	2	3
1	13	012	23
2	012	23	13
3	23	13	012

C^{0123}_{8}	1	2	3
1	12	0123	13
2	0123	12	23
3	13	23	0123

C^{0123}_{10}	1	2	3
1	13	0123	123
2	0123	23	123
3	123	123	012

C^{0123}_{11}	1	2	3
1	13	0123	123
2	0123	23	123
3	123	123	0123

C^{0123}_{13}	1	2	3
1	123	0123	123
2	0123	123	123
3	123	123	012

C^{0123}_{14}	1	2	3
1	123	0123	123
2	0123	123	123
3	123	123	0123

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