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GALOIS THEORY FOR CYLINDRIC ALGEBRAS AND ITS APPLICATIONS

BY

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ABSTRACT. A Galois correspondence between cylindric set algebras and permutation groups is presented in this paper. Moreover, the Galois connection is used to help establish two important algebraic properties for certain classes of finite-dimensional cylindric algebras, namely the amalgamation property and the property that epimorphisms are surjective.

The importance of the amalgamation property (AP for short) in algebraic logic has been recognized for a long time. In [**16**] the connection with the interpolation property of first-order logic is discussed. The positive amalgamation results from the author's thesis [**2**] and their extensions announced in [**3**] are cited in §2.4 of [**16**]. These results are established, with a slight improvement, in §4 below. The key to this study is a lattice anti-isomorphism between subgroups of the symmetric group S_μ and subalgebras of the full set algebra $\mathfrak{A}(\alpha, \mu)$ when $0 < \mu \leqslant \alpha + 1 < \omega$. This Galois theory is developed in §2. It is applied in §3 to show that the algebra $\mathfrak{A}(\alpha, \mu)$, $0 < \mu \leqslant \alpha + 1 < \omega$, is homogeneous and the variety it generates has enough injectives. The announcement [**3**] dealt with the case $\mu \leqslant \alpha$. In this treatment of Galois theory we also include the results of H. Andréka and I. Németi which show that the theory works for $\mu = \alpha + 1$, but for no larger μ.

In §5 the property (ES) that all epimorphisms are surjective is established for certain varieties of CA_α's. It has been shown by I. Németi (see [**14**]) that the ES property for classes of CA_α's is related to Beth's definability result. The question of when the ES property holds was raised in [**8**, p. 311, Problem 10].

To put the results in this paper in perspective, perhaps an additional remark will be useful. Many of the results assert that some property (that depends on α and μ) holds if $\mu \leqslant \alpha + 1 < \omega$. Upon reading a preliminary draft of the paper, Andréka and Németi showed that most of the main results could not be improved in the sense that examples show that the properties fail for $\omega > \mu > \alpha + 1$. In particular, this applies to Theorems 2.2, 2.8, 2.9, 3.7, 3.8(2), Corollaries 3.10, 5.5 and Lemma 5.2(2). Essentially the only result where the value of μ is not known to be the best possible is in Theorem 2.5.

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1. Preliminaries. We primarily use the notation and terminology of [7 and 8], but in order to make the paper reasonably self-contained a quick summary of basic terminology and some unpublished results is given in this section.

The class of all cylindric algebras of dimension α is denoted CA_α and the class of all polyadic equality algebras by PEA_α. The collection $Sb({}^\alpha U)$ of all subsets of ${}^\alpha U$ can be made into a cylindric field of sets in a natural way. This algebra is referred to as the full set algebra with base U and denoted by $\mathfrak{A}(\alpha, U)$. Subalgebras of full set algebras are called set algebras and the class of all such CA_α's is denoted Cs_α. The class of all Cs_α's with base U, where $|U| = \mu$, is denoted ${}_\mu Cs_\alpha$. A generalized cylindric field of sets is a CA_α obtained by relativizing a set algebra to a unit set of the form $\bigcup_{i\in I}{}^\alpha U_i$ where $\{U_i\colon i \in I\}$ is a collection of nonempty pairwise disjoint sets. The U_i's are called subbases and the class of such algebras is denoted by Gs_α. The class of Gs_α's where $|U_i| = \mu$ for each subbase U_i is denoted ${}_\mu Gs_\alpha$. We are particularly interested in the case where $0 < \mu, \alpha < \omega$. In this case, the class $I_\mu Gs_\alpha$ is just the variety generated by $\mathfrak{A}(\mu, \alpha)$.

In a CA_α $\mathfrak{A}$ define

$$C_{(\Gamma)}x = C_{k_1} \cdots C_{k_n}(x) \quad \text{where } \Gamma = \{k_1, \dots, k_n\} \subseteq \alpha.$$

We also let

$$d_\Gamma = \prod_{k,\lambda\in\Gamma} d_{k\lambda} \quad\text{and}\quad \bar{d}(\Gamma \times \Gamma) = \prod_{k,\lambda\in\Gamma;\, k\neq\lambda} - d_{k\lambda} \qquad \text{for } \Gamma \subseteq \alpha.$$

For short, let $\bar{d} = \bar{d}(\alpha \times \alpha)$.

A CA_α $\mathfrak{A}$ has characteristic μ, $0 < \mu < \alpha \cap \omega$, if $\mathfrak{A}$ has a simple minimal subalgebra, $C_{(\mu+1)}\bar{d}((\mu + 1) \times (\mu + 1)) = 0$, and $C_{(\mu)}\bar{d}(\mu \times \mu) = 1$. A CA_α $\mathfrak{A}$, with a simple minimal subalgebra, has *characteristic* 0 if $C_{(\lambda)}\bar{d}(\lambda \times \lambda) \neq 1$ for all $\lambda < (\alpha + 1) \cap \omega$. The class of CA_α's with characteristic μ is denoted ${}_\mu CA_\alpha$. For $0 < \mu < \alpha < \omega$, ${}_\mu CA_\alpha = I_\mu Gs_\alpha$ and all ${}_\mu GS_\alpha$'s with $\mu \geqslant \alpha$ have characteristic 0. The full set algebra $\mathfrak{A}(\alpha, \mu)$ is a simple algebra with characteristic μ if $0 < \mu < \alpha < \omega$ and characteristic 0 if $\alpha \leqslant \mu < \omega$.

The set of all atoms of a CA_α $\mathfrak{B}$ is denoted $At(\mathfrak{B})$. Atoms of simple minimal CA_α's have a nice description (see 2.4.68 of [7]). For a simple minimal CA_α $\mathfrak{A}$ of characteristic μ where $0 \leqslant \mu < \alpha < \omega$, the atoms of $\mathfrak{A}$ consist of all nonzero elements of the form $\bar{d}(\Gamma \times \Gamma) \cdot \prod_{\Delta\in P} d_\Delta$, where P is a partition of α into $\leqslant \mu$ subsets if $\mu \neq 0$, and a partition of α if $\mu = 0$, and Γ is subset of α such that $|\Gamma \cap \Delta| = 1$ for each $\Delta \in P$. The atom associated with a partition P is denoted a_P.

The notion of a PEA_α with characteristic μ is defined in the same way as for CA_α. An unpublished result of Leon Henkin states:

THEOREM 1.1. *Every* PEA_α, $\alpha < \omega$, *of positive characteristic is representable.*

The next result shows that simple PEA_α's and CA_α's with positive characteristic are really the same.

THEOREM 1.2. *For $0 < \mu < \alpha < \omega$ substitution operators can be defined on a simple* ${}_\mu\mathrm{CA}_\alpha$ $\mathfrak{B}$ *such that* $\mathfrak{B}$ *becomes a simple* PEA_α *with characteristic* μ.

The proof of 1.2, which is quite long, will be published separately. The following corollary of 1.1 and 1.2 due to Leon Henkin will be used in 3.8 to help show that certain algebras are injective.

COROLLARY 1.3. *Every* ${}_\mu\mathrm{CA}_\alpha$, *with* $0 < \mu < \alpha < \omega$, *is representable.*

It follows from 1.3 that every simple ${}_\mu\mathrm{CA}_\alpha$, $\mu > 0$, is embeddable in $\mathfrak{A}(\alpha, \mu)$. The substitution operation on $\mathfrak{A}(\alpha, \mu)$ will be denoted $S(\tau)$, i.e., for $\tau \in {}^\alpha\alpha$ and $x \in \mathfrak{A}(\alpha, \mu)$

$$S(\tau)x = \{ f \in {}^\alpha\mu : f\tau \in x \}.$$

The notation $S(\tau)$ is used in place of the standard S_τ to avoid confusion with S_μ which denotes the group of all permutations of μ.

2. Galois theory. For $0 < \mu, \alpha < \omega$ and a group G of permutations on μ, a subalgebra $\mathfrak{A}(\alpha, \mu)_G$ of $\mathfrak{A}(\alpha, \mu)$ is associated with G in the following way. Let $\tilde{G}$ consist of all $\tilde{\sigma}$ for $\sigma \in G$, where $\tilde{\sigma}$ is the automorphism of $\mathfrak{A}(\alpha, \mu)$ defined for each $X \subseteq {}^\alpha\mu$ by $\tilde{\sigma}(X) = \{\sigma y \colon y \in X\}$. It is easily seen that $\tilde{G} \subseteq \mathrm{Aut}\,\mathfrak{A}(\alpha, \mu)$ and the correspondence of σ to $\tilde{\sigma}$ is an isomorphism $S_\mu \cong \mathrm{Aut}\,\mathfrak{A}(\alpha, \mu)$. Let $\mathfrak{A}(\alpha, \mu)_G$ denote the subalgebra of $\mathfrak{A}(\alpha, \mu)$ whose universe consists of all elements fixed by every $\tilde{\sigma} \in \tilde{G}$.

For $\mathfrak{B} \subseteq \mathfrak{A}(\alpha, \mu)$, the *Galois group* of $\mathfrak{B}$, denoted $G(\mathfrak{B})$, is the group of all $\sigma \in S_\mu$ such that $\tilde{\sigma}$ fixes each element of $\mathfrak{B}$.

The following result, whose proof is routine, connects subgroups of S_μ with subalgebras of $\mathfrak{A}(\alpha, \mu)$.

THEOREM 2.1. (1) *If* $H \subseteq G \subseteq S_\mu$, *then* $\mathfrak{A}(\alpha, \mu)_G \subseteq \mathfrak{A}(\alpha, \mu)_H$; *moreover,* $G \subseteq G(\mathfrak{A}(\alpha, \mu)_G)$.

(2) *If* $\mathfrak{B} \subseteq \mathfrak{C} \subseteq \mathfrak{A}(\alpha, \mu)$, *then* $G(\mathfrak{C}) \subseteq G(\mathfrak{B})$; *moreover,* $\mathfrak{B} \subseteq \mathfrak{A}(\alpha, \mu)_{G(\mathfrak{B})}$.

The main goal in this section (Theorem 2.9) is to show that under certain conditions the correspondence in 2.1 becomes a lattice anti-isomorphism. For $\mu \leqslant \alpha$ the proof is related to ideas of Krasner [**10**]; for $\mu = \alpha + 1$, the proof is due to H. Andréka and I. Németi.

For G a subgroup of S_μ and $p \in {}^\alpha\mu$ let $p^G = \{\sigma p \colon \sigma \in G\}$ denote the orbit of p under the action of G on ${}^\alpha\mu$. Clearly, $\mathrm{At}(\mathfrak{A}(\alpha, \mu)_G) = \{ p^G \colon p \in {}^\alpha\mu \}$.

THEOREM 2.2. *For* $0 < \mu \leqslant \alpha + 1 < \omega$ *and a subgroup* G *of* S_μ, $G = G(\mathfrak{A}(\alpha, \mu)_G)$; *thus, the assignment of* $\mathfrak{A}(\alpha, \mu)_G$ *to* G *is one-one.*

PROOF. The conclusion is trivial if $0 < \mu \leqslant 2$; so assume that $2 < \mu$ and $G \subseteq S_\mu$. By 2.1(1), $G \subseteq G(\mathfrak{A}(\alpha, \mu)_G)$. In case $\mu \leqslant \alpha$ choose $p \in {}^\alpha\mu$ such that $\mathrm{Rg}\, p = \mu$. For $\sigma \in G(\mathfrak{A}(\alpha, \mu)_G)$, $\tilde{\sigma}(p^G) = p^G$ so $\sigma p = \tau p$ for some $\tau \in G$. The choice p implies $\sigma = \tau \in G$ and so $G = G(\mathfrak{A}(\alpha, \mu)_G)$, as desired. Now, assume $\mu = \alpha + 1$ and $\sigma \in G(\mathfrak{A}(\alpha, \mu)_G)$. Choose $p \in {}^\alpha\mu$ to be one-one and, as above, find $\tau \in G$ with $\sigma p = \tau p$. Since σ and τ agree on $\mathrm{Rg}\, p$ and $\mu = |\mathrm{Rg}\, p| + 1$, the permutations σ and τ agree everywhere. Hence $\sigma = \tau \in G(\mathfrak{A}(\alpha, \mu)_G)$ as desired.

H. Andréka and I. Németi have shown that Theorem 2.2 is the best possible in the sense that if $\mu \geqslant \alpha + 2 \geqslant 4$, there exist distinct subgroups of S_μ with the same algebra of fixed points. For example, both S_μ and the alternating group on μ have the minimal subalgebra of $\mathfrak{A}(\alpha, \mu)$ as their algebra of fixed points since both are α-fold transitive (because $\mu \geqslant \alpha + 2$).

A function $f \in {}^\alpha\mu$ is called *normal* if $\mu \leqslant \alpha$ and $\mu = \operatorname{Rg} f$ or if $\mu = \alpha + 1$ and f is one-one.

For $f \in {}^\alpha\mu$, $i \in \alpha$, and $u \in \mu$ define $f(i/u) = (f \sim \{(i, f_i)\}) \cup \{(i, u)\}$. For $\mu = \alpha + 1$ and $f \in {}^\alpha\mu$ normal, define $m_i f = f(i/u)$, where u is the unique element of $\mu \sim \operatorname{Rg} f$.

The following facts about functions will be useful.

LEMMA 2.3. *Suppose* $\mu = \alpha + 1$ *and* $f \in {}^\alpha\mu$ *is normal. Then*

(1) *for each* $i < \alpha$, $m_i m_i f = f$;

(2) *for* $\sigma \in S_\mu$ *and* $i < \alpha$, $m_i(\sigma f) = \sigma m_i f$;

(3) *for* $i < j < \alpha$, $f(i/f_j)(j/f_i) = m_i m_j m_i f$ *so* $S((i, j))\{f\} = \{m_i m_j m_i f\}$;

(4) *for* $g \in {}^\alpha\mu$, g *normal, there exists a finite sequence* $i_0, \ldots, i_n$ *of elements of* α *such that* $m_{i_0} \cdots m_{i_n} f = g$.

PROOF. (1)–(3) are clear.

(4) First suppose $\operatorname{Rg} f \neq \operatorname{Rg} g$, say $u \in \mu \sim \operatorname{Rg} f$ and $g_i = u$ for some $i \in \alpha$. Then $\operatorname{Rg} f = \operatorname{Rg} m_i g$ so it remains to consider $\operatorname{Rg} f = \operatorname{Rg} g$. In this case there is a permutation $\tau \in S_\alpha$ such that $g = f\tau$, i.e., $g \in S(\tau^{-1})\{f\}$. Since τ^{-1} is a product of transpositions, the description of g follows from (3).

For a subalgebra $\mathfrak{B}$ of $\mathfrak{A}(\alpha, \mu)$ and $f \in {}^\alpha\mu$, let $X_f^{\mathfrak{B}} = \prod\{b \in B: f \in b\}$. When the algebra $\mathfrak{B}$ is clear from the context we write X_f for short. The X_f's are clearly the atoms of $\mathfrak{B}$.

LEMMA 2.4. *For* $\alpha, \mu < \omega$, $f \in {}^\alpha\mu$, $i < \alpha$ *and* $\mathfrak{B} \subseteq \mathfrak{A}(\alpha, \mu)$:

(1) $C_i X_f = \Sigma\{X_q: q \in C_i\{f\}\}$;

(2) *for* $\mu = \alpha + 1$ *and* $f \in \bar{d}$, $m_i X_f = X_{m_i f}$.

PROOF. (1) Since $f \in X_f$, $X_q \cdot C_i X_f \neq 0$ if $q \in C_i\{f\}$. Hence $X_q \leqslant C_i X_f$ because X_q is an atom and so $C_i X_f \geqslant \Sigma\{X_q: q \in C_i\{f\}\}$. Now suppose $p \in C_i X_f$. Since $p(i/u) \in X_f \cdot C_i X_p$ for some u, $X_f \leqslant C_i X_p$. Thus, $f(i/v) \in X_p$ for some v. Let $q = f(i/v) \in C_i\{f\}$. Since $q \in X_p$, $p \in X_p = X_q$ from which the inclusion $\leqslant$ follows.

(2) For f normal, $\bar{d} \cdot c_i\{f\} = \{f, m_i f\}$ so (1) implies $\bar{d} \cdot c_i X_f = X_f + X_{m_i} f$. If $X_f = \bar{d} \cdot c_i X_f$, then $X_{m_i} f = X_f$ and $m_i X_f = m_i(\bar{d} \cdot c_i X_f) = X_f$ from which $m_i X_f = X_{m_i f}$ follows. If $\bar{d} \cdot c_i X_f$ is not an atom,

$$X_{m_i f} = \bar{d} \cdot c_i X_f \cdot -X_f \leqslant m_i X_f.$$

On the other hand,

$$\bar{d} \cdot c_i X_f = \sum\left\{X_q + X_{m_i q}: q \in X_f\right\}$$

so

$$\bar{d} \cdot c_i X_f \cdot -X_f = \sum \{ X_{m_i q} : q \in X_f \} \geq m_i X_f,$$

which gives $m_i X_f = X_{m_i f}$.

THEOREM 2.5. *For $0 < \mu \leqslant \alpha + 1 < \omega$ every subalgebra $\mathfrak{B}$ of $\mathfrak{A}(\alpha, \mu)$ is generated by a single element, namely, $\mathfrak{B} = \mathrm{Sg}\{X_p\}$ where p is normal.*

PROOF. See I.4.6 of [**8**] for the proof when $\mu \leqslant \alpha$. Assume $\mu = \alpha + 1$. Since the atoms of $\mathfrak{B}$ have the form X_f it suffices to show that if p is normal, then $X_f \in \mathrm{Sg}\{X_p\}$ for every $f \in {}^\alpha\mu$. From 2.4(1),

$$d_{ij} \cdot c_i X_f = \sum \{ x_q : q \in d_{ij} \cdot c_i\{f\} \} = X_{f(i/fj)}$$

so that

(1) $$X_{f(i/fj)} \in \mathrm{Sg}\{X_f\} \quad \text{for every } f \in {}^\alpha\mu.$$

If f is normal, the proof of 2.4(2) shows that $X_{m_i f}$ is either X_f or $\bar{d} \cdot c_i X_f \cdot -X_f$. Hence $X_{m_i f} \in \mathrm{Sg}\{X_f\}$. By 2.3(3) it follows that

(2) $$X_q \in Sg\{X_f\} \text{ whenever } f \in \bar{d} \text{ and } q = f(i/f_j)(j/f_i).$$

Since every $f \in {}^\alpha\mu$ can be obtained from p by a composition of transpositions and replacements, the desired conclusion follows from (1) and (2).

Theorem 2.5 was announced (without proof) for $\mu = \alpha + 1$ in I.4.8 of [**8**] as the result $q(\alpha, \alpha + 1) = 1$. The exact relationship between μ and α for which the conclusion of 2.5 holds is still open. Some partial results are announced in Problem 8 of [**8**, p. 311].

LEMMA 2.6. *Suppose $\mu \leqslant \alpha$ and $p, f \in {}^\alpha\mu$ where p is normal. Let $P = \{\Delta_0, \ldots, \Delta_{\mu-1}\}$ be the partition of α, where $\Delta_j = p^{-1}\{j\}$ for $j < \mu$, and let $Q = \{\Gamma_0, \ldots, \Gamma_{\mu-1}\}$ be a similar partition obtained using f. Choose $\Delta, \Gamma \subseteq \alpha$ such that $\Delta \cap \Delta_j = \{v_j\}$ and $\Gamma \cap \Gamma_j = \{u_j\}$ for each $j \in \mathrm{Rg}\, f$. Let λ_f denote a permutation of α such that $\lambda_f(u_j) = v_j$ for $j \in \mathrm{Rg}\, f$ and let $W = \lambda_f(\Gamma) \subseteq \Delta$. For $f \in a \in \mathrm{At}(\mathfrak{B})$, where $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}(\alpha, \mu)$, define $y_f = a_P \cdot C_{(\alpha \sim W)} S(\lambda_f) a$, where a_P is the atom associated with the partition P (cf. §1). Then $a_Q \cdot C_{(\alpha \sim \Gamma)} S(\lambda_f^{-1}) y_f \leqslant a$.*

PROOF. Suppose $q \in a_Q \cdot C_{(\alpha \sim \Gamma)} S(\lambda_f^{-1}) y_f$. Then there is $q' \in S(\lambda_f^{-1}) y_f$ such that $q \restriction \Gamma = q' \restriction \Gamma$ and for $q(i) = q(u_j)$ for all $i \in \Gamma_j$ (where $j \in \mathrm{Rg}\, f$). Since $q' = h\lambda_f$ for some $h \in y_f$ there exists $h' \in a$ such that $h'\lambda_f^{-1} \restriction W = h \restriction W$. Now for $j \in \mathrm{Rg}\, f$,

$$h'(u_j) = h'\lambda_f^{-1}(v_j) = h(v_j) = h\lambda_f(u_j) = q'(u_j) = q(u_j).$$

Since $h', q \in a_Q$, $q = h' \in a$, which establishes the inclusion.

Actually, $a_Q \cdot C_{(\alpha \sim \Gamma)} S(\lambda_f^{-1}) y_f = a$ holds in Lemma 2.6 but this is not needed below. In 2.8, Lemma 2.6 will be applied with $a = X_p$ where p is normal. We will need to know that $y_f \in B$ whenever $f \in X_p$. For $\mu < \alpha$ we can appeal to 1.2 but a separate argument is needed for $\mu = \alpha$. It does not require much additional work to treat $\mu \leqslant \alpha$. As notation let $X\tau = \{f\tau : f \in X\}$ whenever $X \subseteq {}^\alpha\mu$ and $\tau \in S_\alpha$.

LEMMA 2.7. *Suppose $0 < \mu \leqslant \alpha < \omega$, $p \in {}^{\alpha}\mu$ is normal, $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}(\alpha, \mu)$ and $\tau \in S_\alpha$. Then*

(1) *$X_{p\tau} = X_p\tau$ belongs to* $\mathrm{Sg}\{X_p\}$;

(2) *$S(\tau)X_p$ is an atom of $\mathfrak{B}$.*

PROOF. (1) Since τ is a product of transpositions and, f is normal iff $f\tau$ is normal, it suffices to consider only $\tau = (i, j)$ for $i, j < \alpha$, $i \neq j$. As in 2.6 associate a partition of α, $P = \{\Delta_0, \ldots, \Delta_{\mu-1}\}$, with p. If $p \in d_{ij}$, then $X_p \leqslant d_{ij}$, which yields $q\tau = q$ for all $q \in X_p$. It follows that $X_p\tau = X_p = X_{p\tau}$ in this case.

Now suppose $p \in -d_{ij}$ so that i and j belong to different blocks of the partition P. Without loss of generality, assume $i \in \Delta_0$, $j \in \Delta_1$ and denote $\Delta'_0 = \Delta_0 \sim \{i\}$, $\Delta'_1 = \Delta_1 \sim \{j\}$. Of course, either Δ'_0 or Δ'_1 (or both) may be empty. Consider the partition $R = \{\Delta'_0 \cup \{j\}, \Delta'_1 \cup \{i\}, \Delta_2, \ldots, \Delta_{\mu-1}\}$.

Case 1. $\Delta'_0 = \Delta'_1 = 0$. Note that $P = R$ and $a_R \cdot c_i c_j\{p\} = \{p, p\tau\}$ so Lemma 2.4(1) yields

$$a_R \cdot c_i c_j X_p = a_R \cdot \sum\{X_q : q \in c_i c_j\{p\}\} \tag{3}$$
$$= \sum\{X_q : q \in a_R \cdot c_i c_j\{p\}\} = X_p + X_{p\tau}.$$

If $a_R \cdot c_i c_j X_p$ is an atom, then $X_p = a_R \cdot c_i c_j X_p = X_{p\tau}$, which yields

$$X_p\tau = a_R \cdot \{q\tau : q \in c_i c_j X_p\} = a_R \cdot c_i c_j X_p = X_{p\tau}$$

since $(c_i c_j X_p)\tau = c_i c_j X_p$. Also, note that $X_p\tau \in \mathrm{Sg}\{X_p\}$. Now, if $a_R \cdot c_i c_j X_p$ is not an atom, (3) yields

$$X_{p\tau} = a_R \cdot c_i c_j X_p \cdot -X_p$$

is in $\mathrm{Sg}\{X_p\}$. Also since $q \in X_p$ implies $X_q = X_p$, (3) yields

$$a_R \cdot c_i c_j X_p = \sum\{X_q + X_{q\tau} : q \in X_p\},$$

which gives

$$a_R \cdot c_i c_j X_p \cdot -X_p = \sum\{X_{q\tau} : q \in X_p\} \geqslant X_p.$$

For $f \in a_R \cdot c_i c_j X_p$, $f \restriction \alpha \sim \{i, j\} = g \restriction \alpha \sim \{i, j\}$ for some $g \in X_p$ and $\mu = \{f(i), f(j)\} \cup g^*(\alpha \sim \{i, j\})$. If $f \notin X_p, f \neq g$ so $f = g\tau \in X_p\tau$. Thus,

$$X_{p\tau} = a_R \cdot c_i c_j X_p \cdot -X_p = X_p\tau$$

as desired.

Case 2. $\Delta'_0 \neq 0$ or $\Delta'_1 \neq 0$. Assume, without loss of generality, that $k \in \Delta'_0$. Suppose $f \in a_R \cdot c_i c_j\{g\}$, where $g \in a_p$. Then $f \restriction \alpha \sim \{i, j\} = g \restriction \alpha \sim \{i, j\}$ and $f(j) = f(k) = g(i)$. If $\Delta'_1 \neq 0$, say $l \in \Delta'_1$, then $f(i) = f(l) = g$ so $f = g\tau$; while if $\Delta'_1 = 0$, then $\mu = \{f(i)\} \cup g^*(\alpha \sim \{j\})$, which implies $f(i) = g(j)$, also yielding $f = g\tau$. Thus $a_R \cdot c_i c_j\{g\} \leqslant \{g\tau\}$. The argument above can be reversed to show

$$a_R \cdot c_i c_j\{g\} = \{g\tau\}. \tag{4}$$

Applying (4) when $g \in X_p$ yields $a_R \cdot c_i c_j X_p = X_p \tau$. On the other hand, (4), with $g = p$, and 2.4(1) give

$$a_R \cdot c_i c_j X_p = \sum \{ X_p : q \in a_R \cdot c_i c_j \{ p \} \} = X_{p\tau}.$$

Thus, $X_{p\tau} = X_p \tau$ in Case 2 and part (1) follows.

(2) It is easily seen that $S(\tau) X_p = X_p \tau^{-1}$ which, by (1), is the atom $X_{p\tau^{-1}}$.

THEOREM 2.8. *Suppose $0 < \mu \leqslant \alpha + 1 < \omega$ and $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}(\alpha, \mu)$. Let $G = G(\mathfrak{B})$ and $p \in {}^\alpha\mu$ be normal. Then $X_p = p^G$.*

PROOF. First assume that $\mu = \alpha + 1$. Since $p \in X_p$ and X_p is invariant under G, $p^G \subseteq X_p$. Now suppose $f \in X_p$. Then $f \in \bar{d}$ so we may choose $\alpha \in S_\mu$ with $f = \sigma p$. In order to show $\sigma \in G = G(\mathfrak{B})$ it suffices to show $\tilde{\sigma} X_p \subseteq X_p$ since, by 2.5, X_p generates $\mathfrak{B}$. For $q \in X_p$, 2.3(4) gives $q = m_{i_0} \cdots m_{i_n} p$ ($= \overline{m}p$ for short) for some finite sequence $i_0, \ldots, i_n$ of elements of α. Then $X_p = X_q = X_{\overline{m}p} = mX_p$ by 2.4(2). Hence $\sigma q = \sigma \overline{m} p = \overline{m} \sigma p = \overline{m} f \in \overline{m} X_p = X_p$ (using 2.3(2)) and, thus, $\sigma \in G$ as desired.

Now assume $\mu \leqslant \alpha$. Suppose $p, f \in {}^\alpha\mu$ where p is normal, and associate $P, Q, \Delta, \Gamma, u_j, v_j, \lambda_f$ and W with p and f as in the hypothesis of Lemma 2.6. Note that

$$(1) \qquad p \restriction W = f \lambda_f^{-1} \restriction W.$$

Hence $p \in y_f$ for every f. For the remainder of the proof we assume the atom a in 2.6 is X_p and $f \in X_p$. By 2.7(2), $S(\lambda_f) X_p$ belongs to $\mathfrak{B}$ and, hence, each $y_f \in \mathfrak{B}$. It follows from (1) that $p \in y_f$ for each f, hence

$$(2) \qquad X_p \leqslant y_f \quad \text{for each } f \in X_p.$$

Now, for $\sigma \in S_\mu$,

$$(3) \qquad \sigma \notin G \quad \text{implies} \quad \sigma p \notin X_p.$$

If $\sigma \notin G$, then $\tilde{\sigma}(X_p) \neq X_p$ by 2.5. Thus, $\sigma f \notin X_p$ for some $f \in X_p$. In view of (2), it suffices to show $\sigma p \notin y_f$. Because $\sigma f \in a_Q$ and $\sigma f \restriction \Gamma = \sigma p \lambda_f \restriction \Gamma$ by (1), it is not hard to see that

$$(4) \qquad \sigma f \in a_Q \cdot C_{(\alpha \sim \Gamma)} S(\lambda_f^{-1}) \{ \sigma p \}.$$

Now, if $\sigma p \in y_f$, (4) and 2.6 yield $\sigma f \in X_p$, which contradicts the choice of f. Hence (3) holds.

Finally, since X_p is invariant under G, $p^G \leqslant X_p$. Now suppose $f \in X_p \leqslant a_p$. Then $P = Q$ and there exist $\sigma \in S_\mu$ such that $\Delta_j = \Gamma_{\sigma(j)}$ for $j < \mu$ and $f = \sigma p$. Since $\sigma p \in X_p$, (3) implies $\sigma \in G$. Therefore, $f \in p^G$ and, hence, $X_p \leqslant p^G$.

THEOREM 2.9. *For $0 < \mu \leqslant \alpha + 1 < \omega$ the correspondence in* 2.1 *between subgroups of S_μ and subalgebras of $\mathfrak{A}(\alpha, \mu)$ is a lattice anti-isomorphism.*

PROOF. By 2.2 the correspondence is one-one and by 2.1 it is enough to show that, for every $\mathfrak{B} \subseteq \mathfrak{A}(\alpha, \mu)$, $\mathfrak{B} = \mathfrak{A}(\alpha, \mu)_{G(\mathfrak{B})}$. Let $G = G(\mathfrak{B})$. By 2.1 we know that $\mathfrak{B} \subseteq \mathfrak{A}(\alpha, \mu)_G$ whenever $\mathfrak{B} \subseteq \mathfrak{A}(\alpha, \mu)$. By 2.5 and 2.8, $\mathfrak{B} = \mathrm{Sg}\{ p^G \}$ for any normal

$p \in {}^\alpha\mu$. However, the p^G's are atoms of $\mathfrak{A}(\alpha, \mu)_G$ and, by 2.5, a normal $p \in {}^\alpha\mu$ generates $\mathfrak{A}(\alpha, \mu)_G$. Thus, $\mathfrak{B} = \mathfrak{A}(\alpha, \mu)_G$ as desired.

It was noted following the proof of 2.2 that the assignment of $\mathfrak{A}(\alpha, \mu)_G$ to G was not one-one when $4 \leqslant \alpha + 2 \leqslant \mu < \omega$. Thus, the correspondence in 2.9 will not be an anti-isomorphism in this case. However, one can also ask if, possibly, the correspondence could still be onto. The answer is negative. Andréka and Németi have produced an example of a set algebra $\mathfrak{B} \subseteq \mathfrak{A}(\alpha, \mu)$, again with $\alpha + 2 \leqslant \mu < \omega$, for which $\mathfrak{B} \neq \mathfrak{A}(\alpha, \mu)_{G(\mathfrak{B})}$ and the minimal atom containing each normal p is not an orbit. Thus, the assumption that $\mu \leqslant \alpha + 1$ cannot be removed from either 2.8 or 2.9.

3. Injectives and homogeneous algebras. For reasons that will become apparent in the next section it is useful to know the CA_α-injective algebras. For any class K of similar algebras, an algebra $\mathfrak{A} \in K$ is a *(weak) K-injective* if for every $\mathfrak{B}$, $\mathfrak{C} \in K$, with $\mathfrak{B} \subseteq \mathfrak{C}$, every (monomorphism) homomorphism $h: \mathfrak{B} \to \mathfrak{A}$ extends to a homomorphism $h^+: \mathfrak{C} \to \mathfrak{A}$.

By the following lemma we do not have to distinguish between weak K-injectives and K-injectives when K is a homomorphism closed class of CA_α's.

LEMMA 3.1. *For a class K of similar algebras closed under homomorphic images and possessing the congruence extension property* (CEP), *an algebra is a K-injective if and only if it is a weak K-injective.*

For many classes of algebras there are no nontrivial injective algebras, due to the following simple lemma.

LEMMA 3.2. *For classes K and L, with $K \subseteq L$, where all algebras in K with more than one element have isomorphic minimal subalgebras, there are no nontrivial L-injective algebras in K if K contains simple algebras of arbitrarily high cardinality.*

The lemma applies, in particular, to the case where K is a variety, the notion of "simple algebra in K" is elementary, and K contains an infinite simple algebra. In the case of CA_α's,

COROLLARY 3.3. (i) *There are no nontrivial injective* CA_1*'s.*
(ii) *For $\alpha > 1$ there are no nontrivial injective* CA_α*'s of characteristic* 0.

PROOF. (i) Let $K = L = CA_1$ in 3.2 and note that every BA can be made into a simple CA_1. For (ii), if $1 < \alpha < \omega$ it is not difficult to find an infinite simple algebra (and the notion of "simple algebra" is elementary). For $\alpha \geqslant \omega$, simple Lf_α's of arbitrarily high cardinality can be constructed from models.

Part (i) of 3.3 answers the specific part of Problem 4 in Halmos [**5**]. James S. Johnson observed that 3.2 also implies there are no injective modular lattices (cf. Balbes [**1**]). The following lemma is due to J. Donald Monk.

LEMMA 3.4. *For $\alpha > 1$ every complete discrete* CA_α *is injective.*

PROOF. Since CEP holds in CA_α, by 3.1 it suffices to consider $\mathfrak{B} \subseteq \mathfrak{C}$ and a monomorphism $\mathfrak{B} \rightarrowtail \mathfrak{A}$ where $\mathfrak{A}$ is complete and discrete. If follows that $\mathfrak{B}$ and $\mathfrak{C}$

are discrete and, since the Boolean reduct of $\mathfrak{A}$ is injective, it follows that $\mathfrak{A}$ is injective as a CA_α.

From 3.4 there exist nontrivial CA_α-injectives for all $\alpha > 1$. Below we will see there are nondiscrete ones for $\alpha < \omega$. The next two results reduce the question of injectives to each characteristic.

LEMMA 3.5. *Suppose a* CA_α $\mathfrak{A} \cong P_{\lambda \in \Lambda}\mathfrak{A}_\lambda$ *for some* $\Lambda \subseteq \alpha \cap \omega$ *and each* $\mathfrak{A}_\lambda$ *is a nontrivial* CA_α *with characteristic* λ. *Then* $\mathfrak{A}$ *is a* CA_α*-injective if and only if* $\mathfrak{A}_\lambda$ *is a* CA_α*-injective for all* $\lambda \in \Lambda$.

PROOF. Suppose $\mathfrak{A}$ is a CA_α-injective, $\lambda \in \Lambda$, $\mathfrak{B}$ and $\mathfrak{C}$ are CA_α's, $\mathfrak{B} \subseteq \mathfrak{C}$ and h: $\mathfrak{B} \to \mathfrak{A}_\lambda$ is a homomorphism. Set $\mathfrak{B}_\kappa = \mathfrak{C}_\kappa = \mathfrak{A}_\kappa$ for $\kappa \in \Lambda$, $\kappa \neq \lambda$ and $\mathfrak{B}_\lambda = \mathfrak{B}$, $\mathfrak{C}_\lambda = \mathfrak{C}$. Then $P_{\kappa \in \Lambda}\mathfrak{B}_\kappa \subseteq P_{\kappa \in \Lambda}\mathfrak{C}_\kappa$ and h induces a homomorphism h^+: $P_{\kappa \in \Lambda}\mathfrak{B}_\kappa \to \mathfrak{A}$. Since $\mathfrak{A}$ is CA_α-injective, h^+ extends to a homomorphism k^+: $P_{\kappa \in \Lambda}\mathfrak{C}_\kappa \to \mathfrak{A}$. For $x \in P_{\kappa \in \Lambda}\mathfrak{B}_\kappa$ with $x_\kappa = 0$ for $\kappa \neq \lambda$ and $x_\lambda = 1$, k^+ induces a homomorphism k of $\mathfrak{C} \cong (P_{\kappa \in \Lambda}\mathfrak{C}_\kappa) \upharpoonright x$ into $\mathfrak{A}_\lambda$ which extends h. Thus $\mathfrak{A}$ is a CA_α-injective. The converse is obvious since a product of injectives is injective.

THEOREM 3.6. *Every* CA_α*-injective algebra is a product of injectives of distinct characteristics.*

PROOF. A CA_α-injective is a retract of a complete CA_α and thus complete. The theorem now follows from 3.5 using 2.4.66 and 2.1.33 of [7].

We now turn our attention to CA_α's with a fixed characteristic $\mu \neq 0$ and $\alpha < \omega$. A CA_α $\mathfrak{A}$ is called *homogeneous* if every isomorphism between subalgebras of $\mathfrak{A}$ extends to an automorphism of $\mathfrak{A}$.

As a consequence of the Galois theory result 2.9 there are natural homogeneous CA_α's. Similar results were obtained by Krasner [**10**] for another type of algebraic structure (cf. also [**11, 12**]).

THEOREM 3.7. *For* $0 < \mu \leqslant \alpha + 1 < \omega$, $\mathfrak{A}(\alpha, \mu)$ *is homogeneous.*

PROOF. Suppose f: $\mathfrak{B} \cong \mathfrak{C}$ where $\mathfrak{B} = \mathfrak{A}(\alpha, \mu)_G$ and $\mathfrak{C} = \mathfrak{A}(\alpha, \mu)_H$ according to 2.9. Define $p \in {}^\alpha\mu$ by $p_i = \min\{i, \mu - 1\}$ for $i < \alpha$. Clearly, p is normal, $p^G \in \mathrm{At}(\mathfrak{B})$, so $f(p^G) \in \mathrm{At}(\mathfrak{C})$, say $f(p^G) = q^H$. Choose $\sigma \in {}^\mu\mu$ such that $\sigma(i) = q_i$ for $i < \mu \cap \alpha$ (and $\sigma(\alpha) \in \mu \sim \mathrm{Rg}\, q$ if $\mu = \alpha + 1$). Since p^G, $q^H \leqslant \bar{d}(\Gamma \times \Gamma)$, where $\Gamma = \alpha \cap \mu$, σ is a permutation of μ and thus $\tilde{\sigma}$ is an automorphism of $\mathfrak{A}(\alpha, \mu)$ such that $\tilde{\sigma}(p^G) = f(p^G)$. Since $\mathfrak{B}$ is generated by $\{p^G\}$ (see 2.5 and 2.8) it follows that f is induced by $\tilde{\sigma}$.

The next result shows there are CA_α-injectives with positive characteristic for $\alpha < \omega$. Recall that $I_\mu Gs_\alpha$ is the subvariety of CA_α generated by $\mathfrak{A}(\alpha, \mu)$.

THEOREM 3.8. (1) $\mathfrak{A}(\alpha, \mu)$ *is* CA_α*-injective for* $0 < \mu < \alpha < \omega$.
(2) $\mathfrak{A}(\alpha, \mu)$ *is an* $I_\mu Gs_\alpha$*-injective for* $0 < \mu \leqslant \alpha + 1 < \omega$.

PROOF. (1) In view of 3.1 it suffices to consider

$$\begin{array}{l} \mathfrak{B} \\ \cup\!\mid \\ \mathfrak{A} \overset{f}{\rightarrowtail} \mathfrak{A}(\alpha, \mu) \end{array}$$

where $\mathfrak{A}$, $\mathfrak{B}$ are CA_α's. Since $\mathfrak{A}(\alpha, \mu)$ has characteristic $\mu > 0$, $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the identity $C_{(\mu+1)}\bar{d}((\mu + 1) \times (\mu + 1)) = 0$ and, hence, we may assume $\mathfrak{A}$ and $\mathfrak{B}$ have characteristic μ. For I a maximal ideal of $\mathfrak{B}$, $\mathfrak{B}/I$ is a simple algebra of characteristic μ, hence embeddable in $\mathfrak{A}(\alpha, \mu)$ by the representation result 1.3.

$$\begin{array}{ccccc} \mathfrak{B} & \twoheadrightarrow & \mathfrak{B}/I & \rightarrowtail & \mathfrak{A}(\alpha, \mu) \\ \cup\!\mid & & g \nearrow & & \downarrow \bar{\sigma} \\ \mathfrak{A} & & \overset{f}{\rightarrowtail} & & \mathfrak{A}(\alpha, \mu) \end{array}$$

Let g denote the composite homomorphism restricted to $\mathfrak{A}$. Use 3.7 to obtain an automorphism $\bar{\sigma}$ of $\mathfrak{A}(\alpha, \mu)$ extending fg^{-1}. The outer diagram commutes giving a homomorphism that extends f.

(2) It suffices to consider $\mathfrak{A}, \mathfrak{B} \in I_\mu Gs_\alpha$ such that

$$\begin{array}{l} \mathfrak{B} \\ \cup\!\mid \\ \mathfrak{A} \overset{f}{\rightarrowtail} \mathfrak{A}(\alpha, \mu) \end{array}$$

for some injection f. For any maximal ideal I, $\mathfrak{B}/I$ is simple. Because CA_α's have distributive congruence lattices, Jonsson's Lemma **[9]** implies that $\mathfrak{B}/I$ is embeddable in $\mathfrak{A}(\alpha, \mu)$. The result now follows as in (1) using 3.7.

Theorem 3.8 yields several corollaries.

COROLLARY 3.9. *For $1 < \mu < \alpha < \omega$, a CA_α $\mathfrak{A}$ with characteristic μ is CA_α-injective if and only if $\mathfrak{A}$ is a retract of ${}^I\mathfrak{A}(\alpha, \mu)$ for some $I \neq 0$.*

An algebra with characteristic 1 is injective iff it is complete and discrete. All CA_α-injectives, for $\alpha < \omega$, can be described by combining 3.3(ii), 3.5, and 3.9.

The following will be needed in §4.

COROLLARY 3.10. *For $0 < \mu \leqslant \alpha + 1 < \omega$, $I_\mu Gs_\alpha$ has enough injectives, i.e., every member of $I_\mu Gs_\alpha$ is embeddable in an $I_\mu Gs_\alpha$-injective.*

Using 2.4.66 of [7], for $\alpha < \omega$, a CA_α that satisfies $\bar{d}(\alpha \times \alpha) = 0$ is a product of CA_α's each with some positive characteristic. Thus,

COROLLARY 3.11. *For $1 < \alpha < \omega$, the variety of CA_α's defined by the equation $\bar{d}(\alpha \times \alpha) = 0$ has enough injectives.*

The hypothesis that $\mu \leqslant \alpha + 1$ cannot be dropped from 3.7, 3.8 and 3.10. Andréka and Németi have shown that, for $\omega > \mu \geqslant \alpha + 2$, $\mathfrak{A}(\alpha, \mu)$ is not homogeneous. It follows that the algebra is neither $I_\mu Gs_\alpha$-injective nor can it be embedded in one.

4. Amalgamation property. The amalgamation property for ${}_\mu\mathrm{CA}_\alpha$ with $0 < \mu < \alpha \cap \omega$ was announced in [3]. Since ${}_\mu\mathrm{CA}_\alpha = I_\mu\mathrm{Gs}_\alpha$ for $0 < \mu < \alpha < \omega$, Theorem 4.2 will slightly improve the original result for $\alpha < \omega$.

The following easily proved lemma, due to R. S. Pierce, is a useful tool for establishing the amalgamation property.

LEMMA 4.1. *Suppose K is a class of similar algebras such that:*
(i) *K is closed under isomorphisms and finite products;*
(ii) *K has enough K-injectives.*
Then K has the amalgamation property.

The criteria 4.1 and 3.10 immediately yield:

THEOREM 4.2 (1) *The amalgamation property holds in $I_\mu\mathrm{Gs}_\alpha$ for $0 < \mu \leqslant \alpha + 1 < \omega$.*
(2) *The amalgamation property holds in ${}_\mu\mathrm{CA}_\alpha$ for $0 < \mu < \alpha < \omega$.*

Tracing the arguments involved in the proof of 4.2(1) it is easy to see that for $\mu, \alpha < \omega$, $I_\mu\mathrm{Gs}_\alpha$ has the amalgamation property if and only if $\mathfrak{A}(\alpha, \mu)$ is homogeneous.

The restriction $\alpha < \omega$ in 4.2(2) can be removed using an ultraproduct argument involving reducts of CA_α's similar to the argument in Monk [13].

THEOREM 4.3. *For $1 < \mu < \omega$, ${}_\mu\mathrm{CA}_\alpha$ has the amalgamation property for all $\alpha \geqslant \omega$.*

PROOF. Suppose $\alpha \geqslant \omega$ and $1 < \mu < \omega$. Let L be the set of all finite subsets J of α such that $\mu + 1 \subseteq J$, and for each $J \in L$ choose a one-one function γ_J of $|J|$ onto J that extends the identity on μ. For $\mathfrak{A} \in {}_\mu\mathrm{CA}_\alpha$ let $\mathfrak{A}_{[J]}$ denote the γ_J-reduct of $\mathfrak{A}$ (cf. 2.6.1 of [8]). Then $\mathfrak{A}_{[J]} \in {}_\mu\mathrm{CA}_{|J|}$ for $J \in L$.

Suppose $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ belong to ${}_\mu\mathrm{CA}_\alpha$ such that $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{A} \subseteq \mathfrak{C}$. By 4.2(2), the amalgamation property holds in ${}_\mu\mathrm{CA}_n$ whenever $\mu < n < \omega$. Thus, for each $J \in L$, there exist a ${}_\mu\mathrm{CA}_{|J|}$ $\mathfrak{D}_J$ and monomorphisms g_J, f_J such that the following diagram commutes (δ is the identity embedding).

$$\begin{array}{ccc} \mathfrak{B}_{[J]} & \overset{g_J}{\rightarrowtail} & \mathfrak{D}_J \\ \delta \uparrow & & \uparrow f_J \\ \mathfrak{A}_{[J]} & \overset{\delta}{\rightarrowtail} & \mathfrak{C}_{[J]} \end{array}$$

Let $M_J = \{K \in L : J \subseteq K\} \in \mathrm{Sb}(L)$ for $J \in L$. Then $M = \{M_J : J \in L\}$ generates a proper filter in $\mathrm{Sb}(L)$. Let F denote an ultrafilter in $\mathrm{Sb}(L)$ that extends M.

A structure $\mathfrak{D}$ of the similarity type of CA_α can be defined as follows. We let the universe of $\mathfrak{D}$ equal the ultraproduct $P_{J \in L} D_J / F$ and define the Boolean operations in the usual way. For $\kappa, \lambda < \alpha$ define $d_{\kappa\lambda}^{\mathfrak{D}} = h/F$, where $h \in P_{J \in L} D_J$ is defined by

$$h(J) = \begin{cases} d_{ij}^{\mathfrak{D}_J} & \text{if } \kappa, \lambda \in J \in L, \\ \text{arbitrary} & \text{otherwise,} \end{cases}$$

where $i = \gamma_J^{-1}\kappa$ and $j = \gamma_J^{-1}\lambda$. For $\kappa < \alpha$ and $h \in P_{J \in L} D_J$, define $c_\kappa(h/F) = k/F$, where $k \in P_{J \in L} D_J$ is defined so that when $\kappa \in J$, $k_J = c_{\gamma^{-1}\kappa}(h_J)$.

The structure $\mathfrak{D}$ is a CA_α with characteristic μ. Define $g \in {}^{B}D$ by $g(x) = h/F$, where $h \in P_{J \in L} D_J$ is defined by $h_J = g_J x$ for $J \in L$ and $x \in B$. Similarly, $f \in {}^{C}D$ can be defined. Both f and g are monomorphisms and the diagram

$$\begin{array}{ccc} \mathfrak{B} & \overset{g}{\rightarrowtail} & \mathfrak{D} \\ \delta \uparrow & & \uparrow f \\ \mathfrak{A} & \overset{\delta}{\rightarrowtail} & \mathfrak{C} \end{array}$$

commutes.

The results above have the following corollary. Recall 2.4.66 of [7].

COROLLARY 4.4. *The amalgamation property holds in the following classes*:
(i) *the class of* CA_α*'s that satisfy* $\bar{d}(\alpha \times \alpha) = 0$ *for* $\alpha < \omega$; *and*
(ii) *the class of* CA_α*'s that satisfy* $\bar{d}(\mu \times \mu) = 0$ *for* $0 < \mu < \alpha \cap \omega$.

In the next section we need to know that the amalgamation property holds for the class of simple algebras in various varieties. The next result allows us to apply the previous work.

THEOREM 4.5. *The amalgamation property holds for the class of all simple algebras in a variety* V *of* CA_α*'s if the property holds in* V.

PROOF. Consider simple algebras $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in V$ and monomorphisms $f: \mathfrak{A} \to \mathfrak{B}$ and $g: \mathfrak{A} \to \mathfrak{C}$. Suppose $\mathfrak{D}$ amalgamates $\mathfrak{A} \to \mathfrak{B}$ and $\mathfrak{A} \to \mathfrak{C}$ in V. Then, for a maximal ideal I of $\mathfrak{D}$, $\mathfrak{D}/I$ is simple and it easily follows that $\mathfrak{D}/I$ also amalgamates $f: \mathfrak{A} \to \mathfrak{B}$ and $g: \mathfrak{A} \to \mathfrak{C}$.

5. Epimorphisms. In this section it will be shown that ES (that is, epimorphisms are surjective) holds for certain varieties of CA_α's. We assume throughout that $\alpha < \omega$.

It seems to be a folklore result that epimorphisms are onto in the case of Boolean algebras. An easy way to establish this is, via duality, to show that every monomorphism of Boolean spaces is a one-one map. The same strategy is employed in 5.3 for certain varieties of CA_α's. The appropriate dual of a CA_α is the notion of a reduced α-space introduced in [4]. For $\alpha < \omega$, a reduced α-space is a sheaf $(X, \mathfrak{S})$ of simple CA_α's over a Boolean space X. The duality between CA_α's and α-spaces is reviewed in the next few paragraphs.

For a CA_α $\mathfrak{A}$ let $X(\mathfrak{A})$ denote the space of all maximal ideals of the BA $\mathrm{Zd}(\mathfrak{A})$ of zero-dimensional elements of $\mathfrak{A}$. For $x \in X(\mathfrak{A})$, $\bar{x} = \{a \in A\colon a \leqslant z \text{ for some } z \in x\}$ is an ideal of $\mathfrak{A}$. The *stalk* $\mathfrak{S}_x$ *over* $x \in X(\mathfrak{A})$ is the CA_α $\mathfrak{A}/\bar{x}$ and $\mathfrak{S}(\mathfrak{A}) = \bigcup\{\mathfrak{S}_x\colon x \in X(\mathfrak{A})\}$. For each $a \in A$ define $\sigma_a\colon X(\mathfrak{A}) \to \mathfrak{S}(\mathfrak{A})$ by $\sigma_a(x) = a/\bar{x}$. $\mathfrak{S}(\mathfrak{A})$ is given the smallest topology which makes all σ_a's $(a \in A)$ open. Then $\mathfrak{A}^d = (X(\mathfrak{A}), \mathfrak{S}(\mathfrak{A}))$ is a reduced α-space called the *dual* of $\mathfrak{A}$.

The construction of a CA_α from a reduced α-space uses the sectional functor. Let $\pi\colon \mathfrak{S} \to X$ denote the projection associated with an α-space $(X, \mathfrak{S})$. A function $\sigma\colon X \to \mathfrak{S}$ is a *section* of $(X, \mathfrak{S})$ if $\pi\sigma$ is the identity on X. The set $\Gamma(X, \mathfrak{S})$ of all continuous sections of $(X, \mathfrak{S})$ becomes a CA_α by defining the operations pointwise.

Of particular interest to the discussion here is the duality between CA_α homomorphisms and sheaf morphisms. Given (reduced) α-spaces $(Y, \mathfrak{S})$ and $(X, \mathfrak{T})$, a *sheaf morphism* H: $(Y, \mathfrak{S}) \to (X, \mathfrak{T})$ is a pair $H = (\lambda, \mu)$, where λ is a continuous map $Y \to X$ and μ is a continuous map $Y +_\lambda \mathfrak{T} \to \mathfrak{S}$ such that $\mu_y = \mu(y, -)$ is a homomorphism of $\mathfrak{T}_{\lambda(y)}$ into $\mathfrak{S}_y$. We consider $Y +_\lambda \mathfrak{T} = \{(y, t) \in Y \times \mathfrak{T}: \lambda(y) = \pi t\}$ as a subspace of $Y \times \mathfrak{T}$. A sheaf morphism $(\lambda, \mu) = H$: $(Y, \mathfrak{S}) \to (X, \mathfrak{T})$ of α-spaces produces a CA_α-homomorphism $\Gamma(H)$: $\Gamma(X, \mathfrak{T}) \to \Gamma(Y, \mathfrak{S})$ in the natural way: for $\sigma \in \Gamma(X, \mathfrak{T})$ define $\Gamma(H)\sigma$ by $(\Gamma(H)\sigma)(y) = \mu(y, \sigma(\lambda y))$ for all $y \in Y$. A sheaf morphism h^d: $\mathfrak{B}^d \to \mathfrak{A}^d$ can also be associated with a CA_α homomorphism h: $\mathfrak{A} \to \mathfrak{B}$. Define $h^d = (h^*, h^0)$, where, for $y \in X(\mathfrak{B})$, $h^*(y) = h^{-1} \cap \mathrm{Zd}(\mathfrak{A})$ and, for $y \in X(\mathfrak{B})$ and $a \in A$,

$$h^0\left(h, a/\overline{h^*(y)}\right) = h(a)/\bar{y}.$$

Theorem 1.2 of [4] establishes a dual equivalence between CA_α's (with homomorphisms) and reduced α-spaces (with sheaf morphisms). The next lemma describes sheaf morphisms dual to surjections of CA_α's.

LEMMA 5.1. *Suppose* $\mathfrak{A}, \mathfrak{B} \in \mathrm{CA}_\alpha$ *for* $\alpha < \omega$, h: $\mathfrak{A} \to \mathfrak{B}$, *and* $h^d = (h^*, h^0)$ *maps* $\mathfrak{B}^d \to \mathfrak{A}^d$. *Then h is onto* $\mathfrak{B}$ *iff* (i) h^* *is one-one, and* (ii) *for each* $y \in X(\mathfrak{B})$, $h^0(y, -)$ *is a surjection of the stalk over* $h^*(y)$ *onto the stalk over* y.

PROOF. Assume that (i) and (ii) hold. To simplity notation let $\mathfrak{A}^d = (X, \mathfrak{T})$ and $\mathfrak{B}^d = (Y, \mathfrak{S})$. Because of the natural isomorphisms $\mathfrak{A} \simeq \Gamma(\mathfrak{A}^d)$ and $\mathfrak{B} \simeq \Gamma(\mathfrak{B}^d)$, to show that h is onto it suffices to show that $\Gamma(h^d)$ is onto. Suppose $\sigma \in \Gamma(Y, \mathfrak{S})$. For each $x \in Y$, $h^0(x, -)$ is onto so $h^0(x, t) = \sigma(x)$ for some $t \in \mathfrak{T}_{h^*(x)}$. In fact $t = \tau_x(h^*(x))$ for some $\tau_x \in \Gamma(X, \mathfrak{T})$. Thus, there is a clopen neighborhood N_x of x such that $\Gamma(h^d)(\tau_x)(y) = \sigma(y)$ for all $y \in N_x$. The fact that h^* is one-one and X, Y are Boolean spaces implies that $h^*(N_x)$ is clopen in $h^*(Y)$ and there is a clopen set M_x in X such that $h^*(N_x) = M_x \cap h^*(Y)$. Using compactness (or the partition property of Pierce [15]) there exist a partition of X into clopen subsets $M_0, \ldots, M_{k-1}$ and sections $\tau_i \in \Gamma(M_i, \mathfrak{T})$ such that

$$h^0(y, \tau_i(h^* y)) = \sigma(y)$$

wherever $h^* y \in M_i$ for $i < k$. Defining τ by $\tau(z) = \tau_i(z)$ whenever $z \in M_i$, $i < k$, it follows that $\tau \in \Gamma(X, \mathfrak{T})$ and $\Gamma(h^d)\tau = \sigma$. Thus, $\Gamma(h^d)$ is onto $\Gamma(\mathfrak{B}^d)$ as desired. To prove the converse, first observe that a surjection h, restricted to closed elements, maps $\mathrm{Zd}(\mathfrak{A})$ onto $\mathrm{Zd}(\mathfrak{B})$ whenever $\alpha < \omega$. Thus, (i) follows from BA duality [6, p. 85] while (ii) follows from the definition of h^0.

When the hypothesis $\alpha < \omega$ is removed from 5.1, conditions (i) and (ii) no longer characterize an arbitrary surjection. In general, (i) and (ii) characterize *conformal surjections*, i.e., surjections h: $\mathfrak{A} \to \mathfrak{B}$ such that $h(\mathrm{Zd}(\mathfrak{A})) = \mathrm{Zd}(\mathfrak{B})$. If a $\mathrm{CA}_\alpha \mathfrak{A}$ is regular (in the sense of [4]) every surjection of $\mathfrak{A}$ onto $\mathfrak{B}$ is conformal. In particular, every Lf_α is regular. We leave the details of the more general form of 5.1 to the reader.

The next result shows that "epis are onto" for certain classes of simple CA_α's.

LEMMA 5.2. *The property* ES *holds in the following classes*:
(1) *simple* CA_1'*s*;
(2) ${}_\mu\mathrm{Cs}_\alpha$ *for* $0 < \mu \leqslant \alpha + 1 < \omega$.

PROOF. (1) Simple monadic algebras are Boolean algebras with a closure operator c defined by $cx = 1$ if $x \neq 0$ and $c0 = 0$; thus the conclusion follows from the ES property for BA's.

(2) In order to show that every epimorphism between members of ${}_\mu\mathrm{Cs}_\alpha$ is a surjection, it suffices to consider only inclusion embeddings. Assume $\mathfrak{A} \subset \mathfrak{B} \subseteq \mathfrak{A}(\alpha, \mu)$. By the assumptions on μ and α the Galois correspondence 2.9 gives $\mathfrak{A} = \mathfrak{A}(\alpha, \mu)_G$ and $\mathfrak{B} = \mathfrak{A}(\alpha, \mu)_H$, where $H \subset G$. Thus, there exists an automorphism $\tilde{\sigma}$ of $\mathfrak{A}(\alpha, \mu)$ that fixes each $a \in \mathfrak{A}$ but moves some element of $\mathfrak{B}$. Since $\tilde{\sigma}$ and δ (= identity) agree on $\mathfrak{A}$ but differ on $\mathfrak{B}$, the inclusion is not an epimorphism whenever $\mathfrak{A}$ is a proper subalgebra of $\mathfrak{B}$.

The main result of this section is presented next.

THEOREM 5.3. *Suppose V is a variety of* CA_α'*s*, $\alpha < \omega$, *such that* AP *and* ES *hold in the class of simple members of V. Then* ES *holds in V.*

PROOF. In view of the duality results [4] and Lemma 5.1, it suffices to show that every monomorphism in the category of reduced α-spaces dual to V satisfies properties 5.1(i) and 5.1(ii). We assume $(X, \mathfrak{T})$ and $(Y, \mathfrak{S})$ are sheaves of simple algebras in V over Boolean spaces X and Y and $(h, k) = H\colon (Y, \mathfrak{S}) \to (X, \mathfrak{T})$ is a monomorphism.

Suppose $hx = hy$ for some $x, y \in Y$. Since $\mathfrak{S}_x$, $\mathfrak{S}_y$, and $\mathfrak{T}_{hx}$ are simple algebras in V, the amalgamation property implies there exist a simple $\mathfrak{D}$ in V and monomorphisms $f_x\colon \mathfrak{S}_x \to \mathfrak{D}$ and $f_y\colon \mathfrak{S}_y \to \mathfrak{D}$ such that

$$(1) \qquad f_x \circ k_x = f_y \circ k_y.$$

Consider the sheaf $(1, \mathfrak{D})$ over the one-point space $\{0\} = 1$ and sheaf morphisms

$$H_x = (\lambda_x, \mu)\colon (1, \mathfrak{D}) \to (Y, \mathfrak{S}) \quad \text{and} \quad H_y = (\lambda_y, \nu)\colon (1, \mathfrak{D}) \to (Y, \mathfrak{S}),$$

where $\lambda_x(0) = x$, $\lambda_y(0) = y$, $\mu_0 = f_x$, and $\nu_0 = f_y$. The sheaf $(1, \mathfrak{D})$ is the α-space dual to $\mathfrak{D} \in V$ and (1) implies that $H \circ H_x = H \circ H_y$. Since H is a monomorphism, $H_x = H_y$, from which $x = y$. Thus, h is one-one and 5.1(i) holds.

Fix $x \in Y$. Since ES holds for the simple algebras of V, in order to show that $k_x\colon \mathfrak{T}_{hx} \to \mathfrak{S}_x$ is onto, it suffices to show that k_x is an epi. Hence, suppose $f_0\colon \mathfrak{S}_x \to \mathfrak{D}$ and $f_1\colon \mathfrak{S}_x \to \mathfrak{D}$ for some (simple) $\mathfrak{D}$ such that

$$(2) \qquad f_0 \circ k_x = f_1 \circ k_x.$$

Introduce sheaf morphisms

$$H_0 = (\lambda, \mu)\colon (1, \mathfrak{D}) \to (Y, \mathfrak{S}) \quad \text{and} \quad H_1 = (\lambda, \nu)\colon (1, \mathfrak{D}) \to (Y, \mathfrak{S}),$$

where $\lambda(0) = x$, $\mu_0 = f_0$ and $\nu_0 = f_1$. From (2) it follows that $H \circ H_0 = H \circ H_1$, but since H is a monomorphism this yields $H_0 = H_1$, from which $f_0 = f_1$. Thus, we see that k_x is an epi and hence maps onto $\mathfrak{S}_x$. Since $x \in Y$ is arbitrary, 5.1(ii) holds. This completes the proof of 5.3.

Some consequences of 5.3 are considered below.

COROLLARY 5.4. ES *holds in* CA_1.

PROOF. The amalgamation property for BA's implies AP for the class of simple CA_1's. The result follows by 5.2(1) and 5.3.

The result 5.4 has also been established by I. Sain, who has shown that CA_1 satisfies the strong amalgamation property.

COROLLARY 5.5. ES *holds in* $I_\mu Gs_\alpha$ *where* $0 < \mu \leqslant \alpha + 1 < \omega$. *In particular, it holds in* ${}_\mu CA_\alpha$ *for* $0 < \mu < \alpha < \omega$.

PROOF. By 5.3, 5.2(2), 4.5, and 4.2.

The following corollary was pointed out by H. Andréka and I. Németi.

COROLLARY 5.6. ES *holds in the subvariety of* CA_α*'s defined by* $\bar{d}(\alpha \times \alpha) = 0$ *for* $\alpha < \omega$.

PROOF. The simple algebras of this subvariety are just the simple algebras of characteristic μ, $0 < \mu < \alpha$. Thus, ES for the simple algebras holds by 5.2 and AP holds by 4.4 and 4.5.

In joint work with H. Andréka and I. Németi it has been shown that if $1 < \alpha < \omega$, there exist epimorphisms in CA_α which are not surjective. Moreover, 5.2(2) and 5.5 are best possible in the sense that ES fails in the variety $I_\mu Gs_\alpha$ when $\omega > \mu > \alpha + 1$.

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