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EPIMORPHISMS IN DISCRIMINATOR VARIETIES

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The purpose of this note is to show that epimorphisms are onto maps in certain discriminator varieties. After the proof several corollaries and an example are presented.

THEOREM. *Suppose* V *is a discriminator variety such that the class* V_s *of all simple members of* V *has the amalgamation property* (AP) *and the property* (ES) *that all epimorphisms are onto maps. Then* V *satisfies* ES.

Terminology and notation that is not explained can be found in WERNER [6]. First, we present a lemma that gives a useful characterization of onto homomorphisms in a discriminator variety in terms of maps between the standard sheaves associated with the algebras in the variety. Essentially, we characterize the "sheaf concept" dual to an onto homomorphism.

Recall, from [6], p.41ff, that for $f\colon \underline{A} \to \underline{B}$, maps f^+: Spec$\underline{B}$ → Spec$\underline{A}$ and f^*: Spec$\underline{B}$ + $\underset{\sim}{S}(\underline{A}) \to \underset{\sim}{S}(\underline{B})$ are defined as follows:

(1) $f^+(\Phi) = f^{-1}(\Phi)$ for $\Phi\in$Spec$\underline{B}$, and

(2) $f^*(\Phi, a/f^+(\Phi)) = f(a)/\Phi$ for $\Phi\in$Spec$\underline{B}$ and $a\in A$,

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i.e., $f^*(\Phi,-)$ is the natural embedding of $\underline{A}/f^+(\Phi)$ into $\underline{B}/\Phi$.

The pair (f^+,f^*) is the sheaf morphism associated with f. Proposition 4.8 of [6] gives some useful properties of f^+ and f^*.

LEMMA. *f: $\underline{A} \to \underline{B}$, f is onto $\underline{B}$ if and only if*

(i) *f^+ is one-one, and*

(ii) *for each $\Phi \in \mathrm{Spec}\underline{B}$, $f^*(\Phi,-)$: $\underline{A}/f^+\Phi \to \underline{B}/\Phi$ is onto.*

PROOF. ($\Rightarrow$). First assume that f is onto B and consider (i). Suppose $\Phi_1,\Phi_2 \in \mathrm{Spec}\underline{B}$, $f^+\Phi_1 = f^+\Phi_2$ and $(a,b)\in\Phi_1$. Since f is onto B there exist $x,y\in A$ such that $f(x) = a$ and $f(y) = b$. Hence $(x,y)\in f^+\Phi_1 = f^+\Phi_2$ and so $(a,b)\in\Phi_2$. Therefore $\Phi_1 \subseteq \Phi_2$. By a similar argument $\Phi_2 \subseteq \Phi_1$; so $\Phi_1 = \Phi_2$ and (i) holds. Condition (ii) holds by (vii) of Proposition 4.8 in [6]. ($\Leftarrow$). Suppose f: $\underline{A} \to \underline{B}$ and (i) and (ii) hold. From the sheaf representation of $\underline{B}$ (Theorem 4.9 of [6]), to $b\in\underline{B} \simeq \Gamma\underset{\sim}{S}(\underline{B})$ we have $\hat{b}\in\Gamma\underset{\sim}{S}(\underline{B})$. For each $\Phi\in\mathrm{Spec}B$, $f^*(\Phi,-)$ is onto so there exist $a_\Phi\in A$ so that

$$f(a_\Phi)/\Phi = f^*(\Phi,a_\Phi/f^+\Phi) = b/\Phi.$$

Equivalently, $\widehat{f(a_\Phi)}(\Phi) = \hat{b}(\Phi)$; thus, these two sections agree on some clopen neighborhood U_Φ of Φ. Since f^+ is one-one, continuous there is a clopen $M_\Phi \subseteq \mathrm{Spec}\underline{A}$ such that

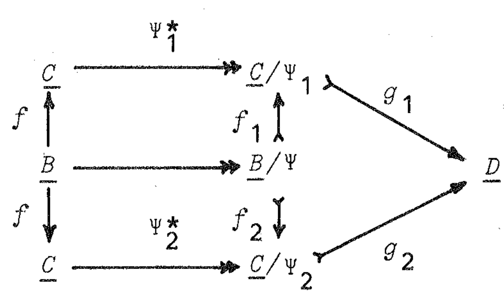
$$f^+(U_\Phi) = M_\Phi \cap f^+(\mathrm{Spec}\underline{B}).$$

By applying compactness to $\{M_\Phi\colon \Phi\in\mathrm{Spec}\underline{B}\}$ and patching together the resulting finite number of a_ϕ's there exist an $a \in A$ such that

$$f^*(\Psi, \hat{a}(f^+\Psi)) = \hat{b}(\Psi)$$

for all $\Psi \in \text{Spec}\underline{B}$. From the representation $\underline{A} \approx \Gamma \underset{\sim}{S}(\underline{A})$ it follows that f is onto $\underline{B}$.

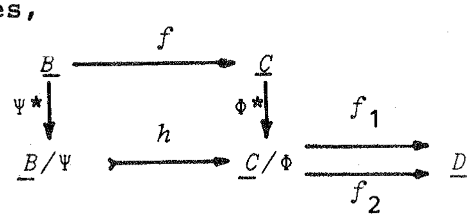
PROOF OF THEOREM. Assume that V is a discriminator variety in which the class V_s of all simple members satisfies AP and ES. To verify ES in V we consider an epimorphism $f\colon B \to C$ in V and show that f satisfies (i) and (ii) of the LEMMA. To establish (i), assume that $\Psi_1, \Psi_2 \in \text{Spec}\underline{C}$ and $f^+\Psi_1 = f^+\Psi_2 = \Psi$. Let $f_i\colon \underline{B}/\Psi \rightarrowtail \underline{C}/\Psi_i$ denote the map induced by f and let $\Psi_i^*\colon \underline{C} \to \underline{C}/\Psi_i$ denote the natural quotient map. By AP in V_s there exist $\underline{D}$ and monomorphisms $g_i\colon \underline{C}/\Psi_i \rightarrowtail \underline{D}$ such that $g_1 \circ f_1 = g_2 \circ f_2$. Thus the following diagram commutes.



Since f is an epimorphism, it follows that $g_1\Psi_1^* = g_2\Psi_2^*$. Now, suppose $(a,b) \in \Psi_1$. Then $\Psi_1^*(a) = \Psi_2^*(b)$, so $g_2\Psi_2^*(a) =$ $= g_1\Psi_1^*(a) = g_1\Psi_1^*(b) = g_2\Psi_2^*(b)$; but g_2 is one-one, so $\Psi_2^*(a) = \Psi_2^*(b)$ which means $(a,b) \in \Psi_2$. Similarly, $\Psi_2 \subseteq \Psi_1$ so

$\Psi_1 = \Psi_2$. Thus (i) holds.

Now consider (ii). Assume $\Phi \in \mathrm{Spec}\underline{C}$ and $\Psi = f^{+}\Phi \in \mathrm{Spec}\underline{B}$. We need to show the natural map $h = f^{*}(\Phi,-): \underline{B}/\Psi \rightarrowtail \underline{C}/\Phi$ is onto. Since ES holds in V_s and B/Ψ and $\underline{C}/\Phi$ are in V_s it suffices to show that h is an epimorphism. Suppose we have $f_1: C/\Phi \to \underline{D}$ and $f_2: \underline{C}/\Phi \to \underline{D}$ in V_s such that $f_1 o h = f_2 o h$. Since f is an epimorphism and the following diagram commutes,



it follows that $f_1 o \Phi^* = f_2 o \Phi^*$. Thus, $f_1 = f_2$ since Φ^* (being onto) is an epimorphism. Hence h is an epimorphism, the proof of (ii) is complete, and the THEOREM follows from the LEMMA.

Propositions 1.9 and 1.11 in [4] show that a variety with AP and ES has the strong amalgamation property (SAP). Using the corollary on p.27 of [6] and the THEOREM we obtain

COROLLARY 1. SAP *holds in every discriminator variety* V *for which* V_s *has* AP *and* ES.

An algebra $\underline{A}$ is called *homogeneous* if each proper inner automorphism of $\underline{A}$ extends to an automorphism of $\underline{A}$. Recall from [6], Theorem 1.14(5), that an algebra $\underline{A}$ is called *demi-primal* if

(i) $HSP(\underline{A})$ is arithmetical,

(ii) all subalgebras of $\underline{A}$ are simple,

(iii) $\underline{A}$ is homogeneous, and

(iv) the subalgebras of $\underline{A}$ are the fixed-point sets of groups of automorphisms.

For $G \subseteq \mathrm{Aut}(\underline{A})$, let $\underline{A}_G$ denote the subalgebra of $\underline{A}$ consisting of all fixed-points of G.

COROLLARY 2. *A residually finite variety V generated by finitely many independent demi-primal algebras has* ES *and, in fact,* SAP.

PROOF. Independent demi-primal algebras are demi-semi-primal so V_s has AP by [6], p.27. To show that ES holds in V_s assume that f: $\underline{C} \to \underline{B}$ is an epimorphism in V_s. Without loss of generality assume that $\underline{C} \subseteq \underline{B} \subseteq \underline{A}$ for some demi-primal $\underline{A} \in V$ and that f is the inclusion map $\underline{C} \to \underline{B}$. By property (iv) in the definition of demi-primal, $\underline{B} = \underline{A}_H$ and $\underline{C} = \underline{A}_G$ for some $H \subseteq G \subseteq \mathrm{Aut}(\underline{A})$. If $\underline{C} \neq \underline{B}$ (i.e., the epimorphism f is not onto), then $H \subset G$ and there exist $\sigma \in G \setminus H$. Thus, σ and id_B both map $\underline{B}$ into $\underline{A}$ and agree on $\underline{C}$. But $\sigma \neq \mathrm{id}_B$ which contradicts the assumption that F is an epimorphism. Hence, $\underline{C} = \underline{B}$ and f is onto. The conclusion now follows from the THEOREM and Corollary 1.

The following special case of Corollary 2 occurs frequently.

COROLLARY 3. *A variety generated by a demi-primal algebra has* ES *and* SAP.

In particular, Corollary 3 yields the result in [2] that the varieties $I_k Gs_n$, where $0 < k \leq n+1 < \omega$, has ES.

The main effort in [2] establishes that certain cylindric set algebras are demi-primal. The variety of all CA_1's is a discriminator variety, however, it is not generated by finitely many finite algebras. Since simple CA_1's are obtained in a standard way from BA's it is easily seen that both AP and ES hold for the class of simple CA_1's. From COROLLARY 1 and the THEOREM we obtain

COROLLARY 4. *CA_1 has* ES *and* SAP.

The result in Corollary 4 has also been obtained by I. SAIN in [5]. Her proof is based on a universal algebraic result for BA's with operators that does not involve a sheaf representation.

For $\underline{B} \subseteq \underline{A}$ let $G_{\underline{B}} = \{\sigma \in \mathrm{Aut}(\underline{A}): \forall x \in \underline{B}(\sigma(x) = x)\}$. The following result provides a method to show that ES fails in a discriminator variety.

PROPOSITION. *Suppose $\underline{A}$ is a finite quasi-primal algebra, $V = \mathrm{HSP}\{\underline{A}\}$ and $\underline{B} \subseteq \underline{A}$. Then the inclusion map $\underline{B} \to \underline{A}$ is an epimorphism if and only if $G_{\underline{B}} = \{\mathrm{id}\}$.*

PROOF. The implication ($\Rightarrow$) follows from the definition of epimorphism. For ($\Leftarrow$), assume that $f: \underline{B} \to \underline{A}$ is not an epimorphism in V. Thus, there exist $\underline{C} \in V$ and homomorphisms $f_0, f_1: \underline{A} \to \underline{C}$ such that $f_0|\underline{B} = f_1|\underline{B}$ and $f_0 \neq f_1$, say $f_0 a \neq f_1 a$ for some $a \in A$. Since $\underline{C}$ is semisimple there is a maximal congruence Φ on $\underline{C}$ such that $(f_0 a, f_1 a) \notin \Phi$. The algebra $\underline{D} = \underline{C}/\Phi$ is simple and the maps $g_i = \Phi^* \circ f_i: \underline{A} \to \underline{D}$ are monomorphisms. Since $\underline{D} \in V_s \subseteq \mathrm{HS}\{\underline{A}\}$, g_0 and g_1 are isomorphisms of $\underline{A}$ onto $\underline{D}$. By the choice of Φ, $g_0 \neq g_1$; so

$f = g_1^{-1} g_0$ is an automorphism of $\underline{A}$, $f \neq \text{id}$. Moreover, $g_i|B = f_i|B$ so f fixes the elements of $\underline{B}$. Hence $f \in G_{\underline{B}} \setminus \{\text{id}\}$ as desired.

The following example can be modified and extended to show that ES fails in several varieties; see [1].

EXAMPLE. Let A denote hte 2-dimensional full cylindric set algebra of all subsets of ${}^2X$ where $X = \{0,1,2,3,4\}$, see HMT[3], p.166. Every permutation of X, acting coordinatewise on ${}^2X$, induces an automorphism of $\underline{A}$ and every automorphism of $\underline{A}$ is obtained in this way. Let a denote the graph of the permutation (01234), b denote the graph of the permutation (02431), d denote the graph of id_X, and $c = {}^2X - (a \cup b \cup d)$. Let $\underline{B}$ denote the subalgebra of $\underline{A}$ generated by $\{a,b\}$. Clearly, $\underline{B}$ is a proper subalgebra of $\underline{A}$ with atoms a,b,c and d. Observe that if the graph of a permutation π is fixed by the automorphism induced by a permutation σ, then σ belongs to the subgroup $\langle\pi\rangle$ generated by π. Since $\langle(01234)\rangle \cap \langle(02431)\rangle = \{\text{id}_X\}$, $G_{\underline{B}} = \{\text{id}\}$. By the PROPOSITION, the inclusion $\underline{B} \to \underline{A}$ is an epimorphism which (as noted) is not onto $\underline{A}$.

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