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COMBINATORIAL TYPES

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Various types of algebraic structures have been used to investigate general properties of abstract languages, for example [HT], [Ha], [Cr], and [AGN]. The general idea behind algebraic logic is to associate a suitable algebraic structure, e.g., a cylindric algebra, with a language (or theory) and to establish properties of the language (or theory) from algebraic, combinatoric, and/or model-theoretic results about the algebraic structures.

The connection between combinatorial schemes and algebraic logic goes back almost twenty-five years to the construction of nonrepresentable relation algebras from projective geometries ([Jo],[Ly]). Constructions of nonrepresentable cylindric algebras also use some type of combinatorial scheme. Monk [Mo1] used the relation algebras derived from projective geometries to show that the class of representable relation algebras is not finitely axiomatizable. A key step in the proof that the class of representable cylindric algebras of dimension n, $n\geq 3$, is not finitely axiomatizable ([Mo2],[Mo3]) also uses *CA*'s constructed from projective geometries. Algebras constructed from other classes of combinatorial systems have been used to show that the classes of group representable cylindric algebras and group representable relation algebras are not finitely axiomatizable ([McK], [Co5]).

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By looking at the proofs of the results cited above it is clear that they are based on certain model-theoretic properties of combinatorial systems. Moreover, the specific results cited are obtained from the combinatorial setting by encoding the combinatorial systems into *CA*'s, *RA*'s, etc. The purpose of this paper is to report on an attempt to isolate the essential combinatorial facts used and to clarify the role of the coding functors. It is hoped that once the situation is clearly understood additional applications of the basic combinatorial results can be made to structures based on other languages and logics.

In section 1 a group-like multivalued system, called a polygroup, is introduced as a combinatorial type. These systems abstract certain properties of combinatorial configurations such as projective geometries and association schemes. A connection is also made with the notion of a data type (in the sense of Scott [Sc]) based on a group. In section 2 a coding of polygroups into CA_3 is briefly studied. This coding is used to establish Monk's result for RCA_3 from a polygroup result in section 3. This particular result was chosen to illustrate the idea of transferring results from combinatorial types to algebraic logic because the nonfinite axiomatizability of RCA_n has already received attention in some computer science work (see e.g., [IL]).

1. COMBINATORIAL TYPES/POLYGROUPS

Various multivalued systems related to algebraic logic have been considered in [Co4] and [Co5]. In this paper, however, we will restrict attention to a group-like system called a polygroup. These systems are easily constructed from combinatorial systems and have the property of being codable in various types of structures studied in algebraic logic.

Definition 1.1. A *polygroup* M is a system $\langle M,\cdot,^{-1},e\rangle$ where $e\in M$, $^{-1}$ is a unary operation on M, $x\cdot y$ is a nonempty subset of

M for each $x,y \in M$, and the following axioms hold for all $x,y,z \in M$:

(i) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

(ii) $x \cdot e = x = e \cdot x$

(iii) $x \in y \cdot z$ implies $y \in x \cdot z^{-1}$ and $z \in y^{-1} \cdot x$

For readability a singleton set is identified with its element; so, for example, x is written in (ii) in place of $\{x\}$. The set-valued operation $\cdot$ on elements extends in the obvious way to subsets: namely, for $A,B \subseteq M$

$$A \cdot B = \bigcup \{a \cdot b : a \in A,\ b \in B\}.$$

Note that (i) is an equality of sets.

Examples of polygroups can be found in [Co4] and [Co5]. We cite a few that are relevant to the present paper.

Example 1.2. Double coset algebras. From the group-like nature of 1.1 it is not surprising that the system of all double cosets of a subgroup H of a group G forms a polygroup. This system is denoted $G//H$.

Example 1.3. Lyndon-Prenowitz algebras ([Pr],[Ly]). Suppose G is a projective geometry with the set of points P and I is a new object, $I \notin P$. A Lyndon-Prenowitz algebra (which is a polygroup) is a system

$$\underline{P}_G = \langle P \cup \{I\}, \cdot, ^{-1}, I \rangle$$

where $x^{-1} = x$ and $I \cdot x = x \cdot I = x$ for all $x \in P \cup \{I\}$ and for $x,y \in P$,

$$x \cdot y = \begin{cases} \overline{xy} \sim \{x,y\} & if\ x \neq y \\ \{I,x\} & if\ x = y \end{cases}$$

[The unique line determined by $x \neq y$ is denoted $\overline{xy}$.]
If the geometry G has dimension 1, i.e., there is only one line X, $\underline{P}_G$ is denoted $\underline{P}_X$ for short.

Example 1.4. Chromatic polygroups. The class of chromatic polygroups play the same role for our combinatorial types that set algebras do in algebraic logic. First we describe a special kind of edge coloring of a complete graph.

Suppose C is a set with a distinguished element $e \in C$ and ζ is an involution of C such that $\zeta(e)=e$. [Think of each element of C except e as a color; e will correspond to the identity relation.] A color scheme is a system $\mathfrak{U} = \langle V, \{R_a : a \in C\}\rangle$ where each $R_a \subseteq V^2$ and

(1) *$\{R_a : a \in C\}$ is a partition of V^2 with*
$R_e = \{(x,x) : x \in V\} = Id$,

(2) $R_{\zeta(a)} = R_a^{\cup}$ for $a \in C$,

(3) *for each* $a \in C$, $x \in V$, $(x,y) \in R_a$ for some $y \in V$,

(4) *for each* $a, b, c \in V$, $R_c \cap (R_a | R_b) \neq 0$ implies $R_c \subseteq R_a | R_b$.

[The converse operation on V^2 is denoted $^{\cup}$ and relation composition is denoted $|$ in the axioms above.] In particular, the notion of a color scheme includes the notion of a homogeneous coherent configuration ([Hi]) and the notion of an association scheme ([BM]). Frequently, for C finite we set $C = \{0, 1, \dots, n\}$ and $e=0$.

Define the system

$$\underline{M}_{\mathfrak{U}} = \langle C, \cdot, ^{-1}, e\rangle$$

where $a^{-1} = \zeta(a)$ for all $a \in C$ and

$$a \cdot b = \{c \in C : R_c \subseteq R_a | R_b\}$$

for all $a, b \in C$. Observe that $a \cdot e = a = e \cdot a$ and $e \in a \cdot a^{-1}$ for all $a \in C$. In fact, $\underline{M}_{\mathfrak{U}}$ is a polygroup called the *color algebra* (or *configuration algebra*) of $\mathfrak{U}$. A polygroup is called *chromatic* if it is isomorphic to the color algebra $\underline{M}_U$ of some scheme U.

In section 3 ultraproducts of polygroups will be needed. These are defined in the usual way. Suppose $\underline{M}_i=\langle M_i, \cdot_i, {}^{\square}i, e_i\rangle$ for $i\in S$ and D is an ultrafilter on S. Then the ultraproduct is

$$\Pi_D \underline{M}_i = \langle \Pi_D M_i, *, {}^{-1}, e\rangle$$

with $e=\{\bar{e}\}_D$ where $\bar{e}\in\Pi M_i$ is given by $\bar{e}(i)=e_i$ for all $i\in S$. For $a,b,c\in\Pi M_i$,

$$\{a\}_D^{-1} = \{b\}_D \quad iff \quad \{i\in S : a_i{}^{\square}i=b_i\}\in D.$$

and

$$\{c\}_D\in\{a\}_D*\{b\}_D \quad iff \quad \{i\in S : c_i\in a_i\cdot_i b_i\}\in D.$$

There are several ways to consider the class of polygroups as a category. For polygroups $\underline{M}=\langle M,\cdot,{}^{-1},e\rangle$ and $\underline{N}=\langle N,\cdot,{}^{-1},e'\rangle$ a function f from M onto N is called a *special morphism*, denoted $f\colon \underline{M} \longrightarrow \underline{N}$, if, for all $x,y,z\in M$,

(i) $fx = e'$ *iff* $x = e$,

(ii) $fz\in fx\cdot fy$ *iff there exist* x',y' *with* $fx'=fx$, $fy'=fy$ *and* $z\in x'\cdot y'$.

Conditions (i) and (ii) imply $(fx)^{-1}=fx^{-1}$.

As in group theory it is often advantageous to describe the image $\underline{N}$ of $\underline{M}$ under f as a "quotient" $\underline{M}//\Theta$ where Θ is the "kernel" of f. An equivalence relation Θ on a polygroup $\underline{M}$ is called a *conjugation* if, for all $x,y\in M$, $(\Theta x)^{-1}=\Theta(x^{-1})$ and $\Theta(xy) \subseteq (\Theta x)(\Theta y)$. A conjugation is *special* is $\Theta e=\{e\}$. Examples of (special) conjugations are abundant; see [Co5]. For a conjugation Θ on $\underline{M}$ the set of Θ-classes forms a system $\underline{M}//\Theta$ that is again a polygroup. In [Co5] it was seen that $G//\Theta$ is chromatic for any conjugation Θ on a group G.

Forms of the ususal homomoprhism and isomorphism theorems hold for polygroups. In particular,

Proposition 1.5. *If* f *is a special morphism of* $\underline{M}$ *onto* $\underline{N}$, *then* $\Theta=\{(x,y)\in M^2:\ fx=fy\}$ *is a special conjugation on* $\underline{M}$ *and* $\underline{N}\cong\underline{M}//\Theta$.

Multivalued systems are related to the theory of semantic domains (alias, continuous posets or data types) developed by Scott [Sc]. As a final example in this section we indicate the connection between polygroups and domains on groups.

Example 1.6. Group domains. For basic information about domains see [Sc]. We suppose G is a set of tokens and $\mathcal{D}$ is a neighborhood system on G. It is convenient to think of G as a set of data and $\mathcal{D}$ as a specifying information about the data. The set $|\mathcal{D}|$ of all filters of $\mathcal{D}$ is partially ordered by an approximability relation $\sqsubseteq$: *for* $x,y\in|\mathcal{D}|$, $x\sqsubseteq y$ *iff* $y\subseteq x$. Maps between sets of tokens give raise to approximable maps between domains. In general, operations on G will induce (partial) operations on $|\mathcal{D}|$.

In order to begin to understand the relationship between operations on G and operations on $|\mathcal{D}|$ it is natural to first consider a simple situation. We suppose $(G,\cdot,e)$ is a group. It is natural to assume there is some relationship between the group operation $\cdot$ and a neighborhood system $\mathcal{D}$ on G. If $\mathcal{D}$ specifies information about the group, it should at least contain all information about which elements can be distinguished.
Given $\mathcal{D}$ consider the equivalence relation $\Theta^{\mathcal{D}}$; namely, for $a,b\in G$

$$a\Theta^{\mathcal{D}}b \ \ iff \ \ \forall X\in\mathcal{D}(a\in X \Longleftrightarrow b\in X).$$

That is, two elements are related if they are indistinguishable given the information provided by $\mathcal{D}$.

If $\mathcal{D}$ contains all the information about distinguishing elements of G, then $\Theta^{\mathcal{D}}$ should satisfy the properties (1),(2) and (3) below:

(1) $x\Theta^{\mathcal{D}}e \Longrightarrow x=e$,

(2) $x\Theta^{\mathcal{D}}y \Longrightarrow x^{-1}\Theta^{\mathcal{D}}y^{-1}$,

(3) $(\Theta^{\mathcal{D}}x)(\Theta^{\mathcal{D}}y)$ *is a union of* $\Theta^{\mathcal{D}}$*-classes (or equivalently,* $\Theta(xy) \subseteq (\Theta y)(\Theta y)$.*)*

Property (1) says that e is a distinguished element while (2) says that if two elements are distinguishable, so are their inverses. Property (3) says that the result of multiplying blocks of indistinguishable elements will not distinguish any additional elements. Of course, (1)-(3) means that $\Theta^{\mathcal{D}}$ is a special conjugation on $(G,\cdot,e)$. How is $G//\Theta^{\mathcal{D}}$ related to $|\mathcal{D}|$ in this case?

Let $t(|\mathcal{D}|)$ denote the set of all total elements of $|\mathcal{D}|$, i.e., the set of maximal filters. In $|\mathcal{D}|$ the "group" operation $\cdot$ may not always produce a total element; for $\bar{x},\bar{y}\in t(|\mathcal{D}|)$ $\bar{x}\cdot\bar{y}$ will usually just approximate a set of total elements. Thus, $t(|\mathcal{D}|)$ may be regarded as a multivalued algebra. The relationship between $G//\Theta^{\mathcal{D}}$ and $t(|\mathcal{D}|)$ is now straightforward.

Proposition *If a group* G *supports a complete domain* $\mathcal{D}$ *(i.e.,* $\mathcal{D}$ *is closed under arbitrary intersections) that satisfies (1), (2) and (3), then the set of total elements* $t(|\mathcal{D}|)$ *forms a polygroup isomorphic to* $G//\Theta^{\mathcal{D}}$.

Property (1) is not essential in the proposition above, i.e., if $\mathcal{D}$ only satisfies (2) and (3), $t(|\mathcal{D}|)$ is still a polygroup and, of course, isomorphic to the quotient $G//\Theta^{\mathcal{D}}$.

Figures 1 and 2 give $|\mathcal{D}|$ and $G//\Theta^{\mathcal{D}}$ for a specific example where $G=Z_5$ and $\mathcal{D}=\{\ \{0\},\ \{1,4\},\ \{2,3\},\ \{0,1,4\},\ \{0,2,3\},\ \{1,2,3,4\},G\}$. Each filter on $\mathcal{D}$ is principal. Let $\bar{I}$ denote the filter generated by I and let $a=\overline{\{1,4\}}$, $b=\overline{\{2,3\}}$ and $e=\overline{\{0\}}$.

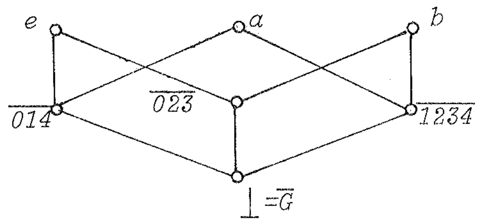


Figure 1
Approximation relation on $|\mathcal{D}|$

*	e	a	b
e	e	a	b
a	a	eb	ab
b	b	ab	ea

Figure 2
Polygroup $G//\Theta$

2. CODING POLYGROUPS INTO CA'S

This section focuses on a functor that codes polygroups into CA_3's. It is a special case of the construction in [Co1] which applies to all multivalued loops.

For a polygroup $\underline{M}=\langle M,\cdot,^{-1},e\rangle$ let $G=\{(a,b,c)\in M^3 : c\in a\cdot b\}$ and $Sb(G)$ denote the collection of all subsets of G. Let $\underline{B}[\underline{M}]$ denote the structure

$$\underline{B}[\underline{M}] = \langle Sb(G),\cup,\cap,\sim,0,G,c_i,d_{ij}\rangle_{i,j<3}$$

where, for $X\subseteq G$,

$$c_i\ X = \{y\in G : y_i=x_i \quad for\ some \quad x\in X\}$$

and

$$d_{ij} = \begin{cases} G & if \quad i=j \\ c_k \{(e,e,e)\} & if \quad \{i,j,k\}=3. \end{cases}$$

It is easy to check that

Proposition 2.1. $\underline{B}[\underline{M}]$ *is a* CA_3.

If $\underline{P}_G$ is the Lyndon-Prenowitz algebra of 1.2, $\underline{B}[\underline{P}_G]$ is (isomorphic to) the CA_3 Monk associated with the geometry G (cf., [Mo2], [Col] .

Proposition 2.2. $\underline{B}[-]$ *is a contravariant functor that weakly preserves ultraproducts. That is,*

(1) $\underline{M} \longrightarrow \underline{N}$ *iff* $\underline{B}[\underline{N}] \longrightarrow \underline{B}[\underline{M}]$ *is a completely additive embedding,*

(2) $\Pi_D\, \underline{B}[\underline{M}_i] \rightarrowtail \underline{B}[\Pi_D\, \underline{M}_i]$.

Proof. (1). By 1.5 we may assume $\underline{N}=\underline{M}//\Theta$ and $\underline{M} \longrightarrow \underline{N}$ is the natural quotient map. Each atom of $\underline{B}[\underline{M}//\Theta]$ corresponds to a triple (B_0,B_1,B_2) in the graph of $*$ on $\underline{M}//\Theta$. To this atom associate

$$\varphi(\{(B_0,B_1,B_2)\}) = G \cap (B_0 \times B_a \times B_2).$$

The function φ on atoms extends to a unique, completely additive embedding of $\underline{B}[\underline{M}//\Theta]$ into $\underline{B}[\underline{M}]$. Conversely, suppose φ is a complete embedding of $\underline{B}[\underline{N}]$ into $\underline{B}[\underline{M}]$. Define $f\colon \underline{M} \longrightarrow \underline{N}$ for $a \in M$, by setting $f(a)$ equal to the unique $b \in N$ such that $(a,e,a) \in \varphi\{(b,e,b)\}$. Since $\varphi(d_{02})=d_{02}$ and φ is completely additive, f is well-defined with $dom f=M$. It is not hard to check that f is a special morphism of $\underline{M}$ onto $\underline{N}$.

(2). Suppose $x \in \Pi \underline{B}[\underline{M}_i]$, $a,b,c \in \Pi M_i$ and D is an ultrafilter on the index set. Define φ on $\Pi_D \underline{B}[\underline{M}_i]$ by

$$(\{a\}_D, \{b\}_D, \{c\}_D) \in \varphi(\{x\}_D \quad iff \quad \{i\colon (a_i,b_i,c_i) \in x_i\} \in D$$

It is straightforward to check that φ is the desired embedding.

A CA_3 is called *integral* if it satisfies

$$\forall x(\ x\neq o \implies c_0c_1\ x=c_0c_2\ x=c_1c_2\ x=1).$$

From 1.1 it is easy to establish that

Lemma 2.3. *For a polygroup* $\underline{M}$, $\underline{B}[\underline{M}]$ *is integral.*

As a consequence of 2.3 each $\underline{B}[\underline{M}]$ is simple. This simplifies the question of when $\underline{B}[\underline{M}]$ is a representable CA_3.

The notion of a cylindric set algebra was extensively studied in [HMTAN]. For a set V and $i,j<3$ let

$$D_{ij} = \{x\in V^3:\ x_i=x_j\}$$

and, for $X\subseteq V^3$,

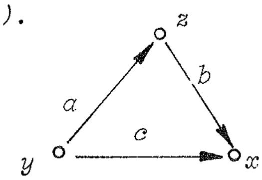
$$C_i\ X = \{y\in V^3:\ (\exists x\in X)(\forall k\neq i)\ x_k=y_k)\}\ .$$

Recall that a 3-dimensional cylindric set algebra with base V (a Cs_3 for short) is a system $\langle A,\cup,\cap,\sim,0,1,C_i,D_{ij}\rangle_{i,j<3}$ where $\langle A,\cup,\cap,\sim,0,1\rangle$ is a Boolean field of subsets of V^3 with unit $1=V^3$ that contains D_{ij} for all $i,j<3$ and is closed under C_i for $i<3$. Any subdirect product of set algebras is called (set) representable (an RCA_3). A simple algebra will be representable if it is isomorphic to a Cs_3. If there is a completely additive isomorphism from a complete atomic CA_3 onto a set algebra whose universe is a complete field of subsets, we say the CA_3 has a *complete representation.*

The next result indicates one reason that chromatic polygroups are important.

Proposition 2.4. *For a polygroup* $\underline{M}$ $\underline{B}[\underline{M}]$ *has a complete representation if, and only if,* $\underline{M}$ *is chromatic.*

Proof. First, assume $\underline{M}$ is chromatic, say $\underline{M}=\underline{M}_U$ for a color scheme $U=\langle V,\{R_a:a\in M\}\rangle$. An embedding φ of $\underline{B}[\underline{M}_U]$ into a set algebra with base V is defined for an atom $\{(a,b,c)\}$ *of* $\underline{B}[\underline{M}_U]$ by $\varphi(\{(a,b,c)\}) = \{(x,y,z)\in V^3: yR_cx,\ zR_bx,\ yR_az\}$ and extended by additive to all of $\underline{B}[\underline{M}_U]$. For a triple of colors (a,b,c) where $c\in a.b$, geometrically φ represents the atom associated with (a,b,c) as the set of triples of points in V which determine a triangle with an edge coloring (a,b,c).



The sets $\varphi(\{(a,b,c)\})$ for $(a,b,c)\in G$ give a partition of V^3 into nonempty subsets, so φ extends to a completely additive Boolean isomorphism. It is straightforward to check that $\varphi(d_{ij}) = D_{ij}$ and that $C_i\varphi(X)=\varphi(c_iX)$ for all $X \subseteq G$. Conversely, suppose φ is a complete representation of $\underline{B}[\underline{M}]$ into a set algebra with base V. For $a\in M$ the relation $R_a \subseteq V$ is defined as

$$R_a = \{(x,y)\in V : (y,x,y)\in\varphi\{(a,e,a)\}\}.$$

Properties (1),(2) and (3) in the definition of a color scheme (see 1.4) easily follow from the fact φ is a completely additive isomorphism and $\varphi(d_{02})=D_{02}$.

Before verifying property (4) of 1.4 we observe.

Lemma *The following are equivalent:*

(i) $c\in a\cdot b$ *in* $\underline{M}$,

(ii) $c_0\{(a,e,a)\}\cap c_1(d_{12}\cdot c_2\{(b,e,b)\})\cap c_2\{(c,e,c)\} \neq 0$,

(iii) $C_0\varphi\{(a,e,a)\}\cap C_1(D_{12}\cdot C_2\varphi\{(b,e,b)\})\cap C_2\varphi\{(c,e,c)\} \neq 0$,

(iv) $R_c\cap(R_a|R_b) \neq 0$.

Now it follows that $\langle V,\{R_a : a\in M\}\rangle$ is a color scheme since (iii) of the lemma is equivalent to

$$C_2(C_0\varphi\{(a,e,a)\}\cap C_1(D_{12}\cap C_2\varphi\{(b,e,b)\}))\geq \varphi\{(c,e,c)\}$$

which is equivalent to $R_c\subseteq R_a|R_b$. The lemma above also shows that $\underline{M}$ is isomorphic to the color algebra of the scheme just constructed.

Several of the constructions and results of this section hold more generally. In particular the construction of $\underline{B}[\underline{M}]$ and the facts 2.1, 2.2 and 2.3 all hold when $\underline{M}$ is a multivalued loop. Moreover, in a sense, multivalued loops allow the complete description of all integral CA_3's. In [Co3] the notion of an adjunction of a CA_3 was introduced and it was shown that every integral CA_3 is embeddable in an adjunction of the complex algebra $\underline{B}[\underline{M}]$ of some multivalued loop $\underline{M}$. Since most of the known nonrepresentable CA_3's are integral the description of all such algebras is fairly important. We leave the development of these remarks to a later publication.

3. NONFINITE AXIOMATIZABILITY

A model-theoretic result is established for the class of chromatic polygroups in 3.5. From this result and the coding properties of $\underline{B}[-]$ Monk's result for RCA_3's will be derived.

First, however, we discuss the language for polygroups. This language contains a constant e, a unary operation $^{-1}$, a binary operation symbol $\cdot$, and binary relation $\in$. Terms and formulas are developed in the usual way except that only variables and constants are allowed for σ in an expression of the form $\sigma \in \tau$. The semantics for the language can be easily understood by regarding the language as halfway between an operation language and a relational language. More precisely, an ordinary binary operation, say f, can be regarded as a 3-place relation F so that $z=f(x,y)$ is equivalent to $F(x,y,z)$. When the operation f is given in the relation form, two additional axioms are added to the logic; namely,

(i) $\forall x, y \exists z \; F(x,y,z)$

(ii) $\forall x,y,z,u (F(x,y,z) \; \& \; F(x,y,u) \Longrightarrow z=u)$.

When a language contains a multivalued operation f, we can also regard f as a ternary relation F, however, in this instance only axiom (i) is added to the logical axioms. In this way, a language with multivalued operations can be regarded as a special relational language. In particular, it is clear how to interpret formulas in structures and that standard model-theoretic arguments can be used. For example, we have

Los' theorem *For any sentence* Φ *in the language of polygroups and any ultraproduct* $\Pi_D \underline{M}_i$ *of polygroups,*

$$\Pi_D \underline{M}_i \models \Phi \quad iff \quad \{i : \underline{M}_i \models \Phi\} \in D.$$

It follows from Los' Theorem that (pseudo) elementary classes (in particular, elementary classes) are closed under ultraproducts. For a pair of pseudo elementary classes K and K' with $K \supseteq K'$, K' is *finitely axiomatizable* over K if there exist a finite set Σ of sentences such that $K'=K \cap Md(\Sigma)$. The following standard consequence of Los' Theorem is basic in a discussion of finite axiomatizability.

Lemma 3.1. *If* K *and* K' *are pseudo elementary classes,* $K \supseteq K'$, *such that* K' *is finitely axiomatizable over* K, *then* $K \sim K'$ *(the complement of* K' *in* K*) is closed under ultraproducts.*

It is worthwhile to note which classes introduced in section 1 are (pseudo) elementary. Obviously the class of polygroups is elementary.

Lemma 3.2. *(1) The class of chromatic polygroups is a pseudo-elementary class.*
(2) The class of Lyndon-Prenowitz algebras P_G *for which* G *has dimension 1 is an elementary class.*

Proof. (1) A chromatic polygroup is a reduct of a system $\langle M, \cdot, ^{-1}, e, V, R\rangle$ for which $\langle M, \cdot, ^{-1}, e\rangle$ is a polygroup and $R: M \longrightarrow Sb(V^2)$ has the properties that $\langle V, \{R(a): a \in M\}\rangle$ is a color scheme and for all $a, b, c \in M$,

$$c \in a \cdot b \quad if \quad R(c) \subseteq R(a) | R(b).$$

(2). If a geometry has dimension 1 (i.e., there is only one line X), $\overline{xy}=X$ whenever $x \neq y$. Using this fact the definition of the operations of $\underline{P}_X$ in Example 1.3 are clearly elementary.

The following important result about the algebras $\underline{P}_G$ was proved by Lyndon [Ly] who expressed the ideas in the language of relation algebras. The present terminology makes the ideas clearer.

Theorem 3.3. *A geometry G is embeddable as a hyperplane in a geometry of one higher dimension iff $\underline{P}_G$ is chromatic.*

Proof. Suppose G, whose point set is P, is embeddable in a geometry with a point set H of one higher dimension. A color scheme on $V=H \sim P$ is constructed using the set $\{I\} \cup P$ of colors by setting $R_I=\{(x,x) : x \in V\}$ and, for $p \in P$,

$$R_p = \{(x,y) \in V^2 : \ x \neq y \ \& \ p \in \overline{xy}\}.$$

Properties (1),(2) and (3) in 1.4 are clear; for (4) the hypothesis $R_c \cap (R_a | R_b) \neq 0$ implies that a,b and c are collinear from which (4) follows by the Pasch axiom. It is also straightforward to show that the color algebra of the scheme above is isomorphic to $\underline{P}_G$. For the converse, we assume that $\underline{P}_G$ is isomorphic to the color algebra of a scheme $\langle V, \{R_p : p \in P \cup \{I\}\}\rangle$ where $P \cap V = 0$ and build a geometry H on $H=P \cup V$. The lines of H are either

(i) lines of G, or

(ii) sets $L(x,p)=\{x,p\} \cup \{y \in V : (x,y) \in R_p\}$ where $x \in V$ and $p \in P$.

The verification that H is a projective geometry with G as a hyperplane is left to the reader.

Corollary 3.4. (Lyndon) *(1). If $\dim G>2$ or G is a Desarguesian plane, then $\underline{P}_G$ is chromatic.*
(2). $\underline{P}_G$ is not chromatic if G is a nonDesarguessian plane.
(3). For G (with dimension 1) a line containing $n+1$ points, $\underline{P}_G$ is chromatic iff there exist a projective plane with order n.

Theorem 3.5. *The class of nonchromatic polygroups is not closed under ultraproducts.*

Proof. By the Bruck-Ryser theorem [BR] there exist an infinite set of positive integers $n_1<n_2<\dots$ for which there is no projective plane with order n_i. For each n_i in this set let $\underline{P}_i$ denote the Lyndon-Prenowitz algebra of the 1-dimensional geometry with n_i+1 points on a line. By 3.4, P_i is not chromatic. However, for any nonprincipal ultrafilter D on the set $\{n_1,n_2,\dots\}$ the ultraproduct $\Pi_D \underline{P}_i$ is isomorphic to $\underline{P}_X$ for an infinite set X by 3.2(2). Thus, $\Pi_D P_i$ is chromatic by 3.4 since projective planes of every infinite cardinality exist.

The result above is essentially to combinatorial/model-theoretic nucleus of Monk's work [Mo2]. It leads immediately to the promised result for RCA_3.

Theorem 3.6. *([Mo2]) RCA_3 is not finitely axiomatizable.*

Proof. By 3.1 it suffices to show the complement of RCA_3 is not closed under ultraproducts. Using 3.5 there exist a collection of nonchromatic finite polygroups P_i and a nonprincipal ultrafilter D such that $\Pi_D P_i$ is chromatic. By 2.2 and 2.4 $\Pi_D \underline{B}[\underline{P}_i]$ is in RCA while 2.4 implies that each $\underline{B}[\underline{P}_i]$ is not representable.

4. CONCLUDING REMARKS

In the previous section we saw that the nonfinite axiomatizability of RCA_3 was obtained from a model-theoretic property of chromatic polygroups using a coding $\underline{B}[-]$. Other results of the same type may be obtained.

In [Co5] a functor $\underline{A}[-]$ from polygroups into relation algebras *(RA's)* was developed with properties analogous to 2.1, 2.2, 2.3 and 2.4. By an argument similar to the one given in 3.6, using $\underline{A}[-]$, Theorem 3.5 yields the following result of Monk [Mo1].

Theorem 4.1. *The class RRA of representable relation algebras is not finitely axiomatizable.*

Let *Q(Group)* denote the class of all polygroups isomorphic to quotients $G//\Theta$ of a group G modulo a conjugation Θ and let Q_s*(Group)* denote the subclass of *Q(Group)* obtained by using special conjugations (see [Co5]). An analysis of the argument in [McK] yields.

Theorem 4.2. *There is an ultraproduct of finite polygroups from* $Q(Group) \sim Q_s(Group)$ *that is in* $Q_s(Group)$.

McKenzie's result [McK] that *GRA* is not finitely axiomatizable is a consequence of 4.2 uisng $\underline{A}$[-] while the result announced in [Co2] follows from 4.2 using $\underline{B}$[-] as in the proof of 3.6. In view of the connection (see 1.6) between system of the type $G//\Theta$ and group domains it is likely that 4.2 will be useful in the study of axiomatizability of certain domains.

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