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THE ELEMENTARY THEORY OF INTERVAL REAL NUMBERS

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0. Introduction and Preliminaries

A model $\mathscr{I}(\mathsf{R})$ for the "real numbers" useful in numerical analysis was introduced by R. E. MOORE [5]. A few years later D. SCOTT [8] observed that the structure underlying $\mathscr{I}(\mathsf{R})$ could be treated as a data type (or continuous poset) in which the interval operations are computable. It is natural to ask for characterizations of $\mathscr{I}(\mathsf{R})$. In section 1 axioms are given for its first-order (elementary) theory. A decision procedure for the theory is given in section 2; basically a statement is translated into the theory of real-closed fields for which TARSKI's quantifier elimination method ([10], [3]) is available. The axioms used in section 1 are somewhat artificial. In the last section a more natural set of axioms is developed (regarding the models as "data types"). While the proposed notion of a partial real closed field is still subject to some artificiality, it can certainly serve as a basis for additional axiomatic, algebraic, and model-theoretic investigations of domains like $\mathscr{I}(\mathsf{R})$.

Let $I(\mathsf{R})$ denote the set of all compact real intervals. For any operation $* \in \{+, \cdot, -\}$ and $A, B \in I(\mathsf{R})$ define $A * B = \{a * b\colon a \in A,\ b \in B\}$. We identify one element intervals with real numbers, i.e., $[a, a] = a$, so R is embedded in $I(\mathsf{R})$ in such a way that all operations are preserved. There are several ways to introduce an order relation on $I(\mathsf{R})$. We define: $[a, b] < [c, d]$ iff $b < c$. Also, for $A, B \in I(\mathsf{R})$: $A \leqq B$ iff $A < B$ or $A = B$. Another ordering that plays an important role is the relation $\sqsubseteq$ that means "is approximate to". For $A, B \in I(\mathsf{R})$: $A \sqsubseteq B$ iff $A \supseteq B$. The idea is that smaller interval contains more information. The basic model for interval analysis introduced and studied in [5] and [6] is the system $\mathscr{I}(\mathsf{R}) = \langle I(\mathsf{R}), \sqsubseteq, +, -, \cdot, 0, 1, < \rangle$. $\langle I(\mathsf{R}), \sqsubseteq \rangle$ is essentially a domain in the sense of SCOTT ([8], [9]); the difference is that SCOTT's continuous poset contains one more element — the unique "bottom" element $\perp$. The maximal (alias "perfect", alias "total") elements of the domain are precisely the ordinary real numbers and $+$, $-$, $\cdot$ and $<$ are all computable.

We will be discussing $\mathscr{I}(\mathsf{R})$ in two first-order languages with equality. The first has $+, \cdot, -, 0, 1$, and $<$ as non-logical symbols while the second language contains, in addition, the symbol $\sqsubseteq$. Of course the languages may be enriched by introducing defined symbols. For example, $x \leqq y$, $x = \min\{y, z, u, v\}$, and $\boldsymbol{R}x$. These are defined as:

$$
\begin{array}{lll}
x \leqq y & \text{iff} & x < y \text{ or } x = y; \\
x = \min\{y, z, u, v\} & \text{iff} & (x = y \mathbin{\&} y \leqq z \mathbin{\&} y \leqq u \mathbin{\&} y \leqq v) \vee \ldots \vee \\
 & & (x = v \mathbin{\&} v \leqq y \mathbin{\&} v \leqq z \mathbin{\&} v \leqq u); \\
\boldsymbol{R}x & \text{iff} & x + (-x) = 0.
\end{array}
$$

Other standard notions such as $x = \max\{y, z, u, v\}$ will be introduced without mention.

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For a system $\mathscr{M} = \langle M, +, -, \cdot, 0, 1, < \rangle$ where $\boldsymbol{R}0$ and $\boldsymbol{R}1$ hold, let $\boldsymbol{R}^{(\mathscr{M})}$ denote the relativized system $\langle \boldsymbol{R}^{\mathscr{M}}, +', -', \cdot', 0, 1, <' \rangle$ where $*'$ denotes the restriction of the operation $*$ to $\boldsymbol{R}^{\mathscr{M}} = \{a \in M: \boldsymbol{R}a$ holds in $\mathscr{M}\}$. For example, observe that $\boldsymbol{R}^{(\mathscr{I}(\mathsf{R}))} = \mathscr{R}$, the ordinary field of real numbers.

1. Axioms for the Theory of Interval Real Numbers

The statements we use to characterize the elementary theory of $\mathscr{I}(\mathsf{R})$ will consist of two collections of axioms. First, let Σ_{R} be any set of statements with the property that, for any model $\mathscr{M}$ of $\Sigma_{\mathsf{R}'}$, $\boldsymbol{R}^{(\mathscr{M})}$ is elementarily equivalent to R, the ordinary real numbers. One way to produce Σ_{R} would be to relativize one of the standard sets of axioms for real closed fields to the predicate $\boldsymbol{R}x$ (see, e.g., [2], [3], or [4]). In section 3 an alternate set of axioms is given.

The statements $(\mathrm{I}_1)-(\mathrm{I}_7)$ given below are called interval axioms. Let Σ_I denote the set of these axioms.

(I_1) $\quad \forall x\, \exists \underline{x}\, \exists \bar{x}[\boldsymbol{R}\underline{x}\ \&\ \boldsymbol{R}\bar{x}\ \&\ \forall t((x < t \leftrightarrow \bar{x} < t)\ \&\ (t < x \leftrightarrow t < \underline{x}))]$.

Before preceding, observe

Lemma 1. *$\Sigma_{\mathsf{R}} \cup \{(\mathrm{I}_1)\}$ implies (1) both $\bar{x}$ and $\underline{x}$ are unique, and (2) $\underline{x} \leqq \bar{x}$.*

Proof. Immediate from trichotomy law and (I_1).

Lemma 1 justifies the addition of unary operations $\underline{x}$ and $\bar{x}$ to our language. They will be used freely in the remaining axioms.

(I_2) $\quad \forall y\, \forall z[\boldsymbol{R}y\ \&\ \boldsymbol{R}z\ \&\ y \leqq z \to \exists x(\underline{x} = y\ \&\ \bar{x} = z)]$,

(I_3) $\quad \forall x\, \forall y[\underline{x} = \underline{y}\ \&\ \bar{x} = \bar{y} \to x = y]$.

Axioms (I_2) and (I_3) immediately yield

Lemma 2. *The x asserted to exist in (I_2) is unique. For each y and z with $y \leqq z$ denote this unique element by $y{:}z$.*

To see that the axioms so far describe every element as a pair of "real numbers" we obtain

Lemma 3. *$\Sigma_{\mathsf{R}} \cup \{(\mathrm{I}_1), (\mathrm{I}_2), (\mathrm{I}_3)\}$ implies, for all x, (1) $x = \underline{x} : \bar{x}$, and (2) if $\boldsymbol{R}x$ then $\underline{x} = x$ and $\bar{x} = x$.*

Proof. (1) By Lemma 1 and (I_2), $z = \underline{x} : \bar{x}$ exist where $\underline{z} = \underline{x}$ and $\bar{z} = \bar{x}$; by (I_3), $z = x$. (2) By (I_1), $\boldsymbol{R}\underline{x}$ holds so, assuming $\boldsymbol{R}x$, the trichotomy law yields exactly one of $x = \underline{x}$, $x < \underline{x}$, or $\underline{x} < x$. It is impossible for $\underline{x} < x$ to hold since it implies (by (I_1)) $\underline{x} < \underline{x}$ contrary to Σ_{R}. Also, $x < \underline{x}$ is impossible since it implies (by (I_1)) that $\bar{x} < \underline{x}$ which violates Lemma 1. (2). Thus, $x = \underline{x}$. Similarly $x = \bar{x}$.

The last four axioms $(\mathrm{I}_4)-(\mathrm{I}_7)$ describe the appropriate relations.

(I_4) $\quad \forall x, y, z[x + y = z \leftrightarrow \underline{x} + \underline{y} = \underline{z}\ \&\ \bar{x} + \bar{y} = \bar{z}]$,

(I_5) $\quad \forall x, y, z[x \cdot y = z \leftrightarrow \underline{z} = \min\{\underline{x}\underline{y}, \underline{x}\bar{y}, \bar{x}\underline{y}, \bar{x}\bar{y}\}\ \&\ \bar{z} = \max\{\underline{x}\underline{y}, \underline{x}\bar{y}, \bar{x}\underline{y}, \bar{x}\bar{y}\}]$,

(I_6) $\quad \forall x, y[-x = y \leftrightarrow -\underline{x} = \bar{y}\ \&\ -\bar{x} = \underline{y}]$,

(I_7) $\quad \forall x, y[x < y \leftrightarrow \bar{x} < \underline{y}]$.

This completes the axioms in Σ_I. Models of $\Sigma_{\mathsf{R}} \cup \Sigma_I$ will be called *interval real number systems*.

Theorem 1. *The axioms for interval real number systems characterize the first-order theory of* $\mathscr{I}(\mathsf{R})$.

Proof. Clearly $\mathscr{I}(\mathsf{R})$ satisfies $\Sigma_{\mathsf{R}} \cup \Sigma_I$. For an arbitrary model $\mathscr{M}$ of $\Sigma_{\mathsf{R}} \cup \Sigma_I$ it suffices to show that $\mathscr{M}$ and $\mathscr{I}(\mathsf{R})$ have isomorphic ultrapowers (cf. [2, Chapter 6]). Since $\mathscr{M}$ satisfies Σ_{R}, $\boldsymbol{R}^{(\mathscr{M})} \underset{\text{e.e.}}{\equiv} \mathscr{R}$. Hence, there exist ultrafilters F and G on sets I and J respectively such that

$$\varphi\colon (\boldsymbol{R}^{(\mathscr{M})})^I_F \cong \mathscr{R}^I_G$$

for some isomorphism φ. Observe that $\boldsymbol{R}^{(\mathscr{M}_F{}^I)} = (\boldsymbol{R}^{(\mathscr{M})})^I_F$ and $\boldsymbol{R}^{(\mathscr{I}(\mathsf{R})_G{}^J)} = \mathscr{R}^J_G$ so the "real numbers" in $\mathscr{M}^I_F$ and $\mathscr{I}(\mathsf{R})^I_G$ are isomorphic by φ. Extend φ to $\varphi^+\colon \mathscr{M}^I_F \to \mathscr{I}(\mathsf{R})^J_G$ by

$$\varphi^+(x) = \varphi(\underline{x}) : \varphi(\bar{x})$$

for all $x \in M^I_F$. Lemma 3 shows that φ^+ extends φ. Suppose $\varphi^+(x) = \varphi^+(y)$. Then $\varphi(\underline{x}) = \varphi^+(x) = \varphi^+(y) = \varphi(\underline{y})$ and similarly $\varphi(\bar{x}) = \varphi(\bar{y})$. Since φ is one-one, these equations and (I_3) imply $x = y$; so φ^+ is one-one. Now, given $w \in \boldsymbol{I}(\mathsf{R})^J_G$ there exist $\underline{w}, \bar{w} \in \boldsymbol{R}^J_G$ such that $\underline{w} \leqq \bar{w}$. Let $a = \varphi^{-1}(\underline{w})$ and $b = \varphi^{-1}(\bar{w})$. Then $a \leqq b$ by the isomorphism property of φ. Let $z = a : b$. Then an easy computation shows $\varphi^+(z) = w$; so φ^+ is onto. Finally, φ preserves $+$, $-$, $\cdot$, and $<$ for "real numbers" so axioms $(\mathrm{I}_4)-(\mathrm{I}_7)$ show that φ^+ is an isomorphism of $\mathscr{M}^I_F$ onto $\mathscr{I}(\mathsf{R})^J_G$.

2. Decidability

It follows from Theorem 1 that the theory of $\mathscr{I}(\mathsf{R})$ (that is, the theory of interval real number systems) is decidable since it is complete and axiomatizable. Another proof of this result is outlined in this section. The ideas is to translate a statement about interval real number systems into one about $\mathscr{R}$ (or real closed fields) which may be analyzed using Tarski's elimination of quantifier method. Some consequences of this process are mentioned at the end of the section.

To translate the theory of $\mathscr{I}(\mathsf{R})$ into the theory of $\mathscr{R}$ first introduce two new (real) variables $\underline{v}$ and $\bar{v}$ with each variable v (that ranges over $\boldsymbol{I}(\mathsf{R})$). We must make sure all the variables $\underline{v}$, $\bar{v}$, $\underline{w}$ and $\bar{w}$ are distinct whenever v and w are distinct variables. A formula φ^I in the languages of $\mathscr{R}$ is inductively associated with each formula φ in the interval language in the following way. For atomic formulas,

$(u = v)^I$ is $(\underline{u} = \underline{v} \,\&\, \bar{u} = \bar{v})$,

$(u < v)^I$ is $(\underline{u} < \underline{v})$,

$(u = 0)^I$ is $(\underline{u} = 0 \,\&\, \bar{u} = 0)$, $\quad (u = 1)^I$ is $(\underline{u} = 1 \,\&\, \bar{u} = 1)$,

$(u = -v)^I$ is $(-\underline{v} = \bar{u} \,\&\, -\bar{v} = \underline{u})$,

$(u = v + w)^I$ is $(\underline{u} = \underline{v} + \underline{w} \,\&\, \bar{u} = \bar{v} + \bar{w})$,

$(u = v \cdot w)^I$ is $(\underline{u} = \min(\underline{v}\underline{w}, \underline{v}\bar{w}, \bar{v}\underline{w}, \bar{v}\bar{w}) \,\&\, (\bar{u} = \max(\underline{v}\underline{w}, \underline{v}\bar{w}, \bar{v}\underline{w}, \bar{v}\bar{w}))$.

By induction,

$(\neg\varphi)^I$ is $\neg(\varphi^I)$,

$(\varphi \vee \psi)^I$ is $(\varphi^I \vee \psi^I)$, $\quad (\varphi \,\&\, \psi)^I$ is $(\varphi^I \,\&\, \psi^I)$,

$(\exists v\varphi)^I$ is $\exists \underline{v}\, \exists \bar{v} (\underline{v} \leqq \bar{v} \,\&\, \varphi^I)$.

Now, by an easy induction argument we obtain

Lemma 4. *For any formula* φ *in the language of* $\mathscr{I}(\mathsf{R})$, *any model* $\mathscr{M}$ *of* $\Sigma_{\mathsf{R}} \cup \Sigma_I$, *and any sequence* $X_0, X_1, \ldots$ *in* M *with* $X_i = \underline{x}_i : \bar{x}_i$ *with* $\underline{x}_i, \bar{x}_i \in \boldsymbol{R}^{\mathscr{M}}$ $(i < \omega)$,

(1) φ^I *is a formula in the language of* $\mathscr{R}$ *with the same quantifier complexity as* φ *and having* $2n$ *free variables whenever* φ *has* n *free variables,*

(2) $\langle X_0, X_1, \ldots \rangle$ *satisfies* φ *in* $\mathscr{M}$ *iff* $\langle \underline{x}_0, \bar{x}_0, \underline{x}_1, \bar{x}_1, \ldots \rangle$ *satisfies* φ^I *in* $\boldsymbol{R}^{(\mathscr{M})}$.

From Lemma 4 it immediately follows that, for a sentence φ,

(3) φ *is true in* $\mathscr{I}(\mathsf{R})$ *iff* φ^I *is true in* $\mathscr{R}$.

(That is, φ is provable from $\Sigma_{\mathsf{R}} \cup \Sigma_I$ iff φ^I is true in all real closed fields.) The following theorem is an immediate consequence of (3) and TARSKI's elimination of quantifier argument for real closed fields [10] (see also [3]).

Theorem 2. *The theory of interval real number systems is decidable.*

The decision process above is fairly tedious to carry out. As an example of a very easy special case we apply the process to the interval formula $\exists X(A + X = B)$. Translating into $\mathscr{R}$ we see the formula is equivalent to $\underline{b} - \underline{a} \leq \bar{b} - \bar{a}$. This is essentially RATSCHEK's result [7] that an interval equation $A + X = B$ has a solution iff the length of A is at most the length of B.

The elimination of quantifier procedure for real closed fields and Lemma 4 yield

Theorem 3. *The theory of interval real number systems is model-complete.*

As an application of model-completeness a Nullstellensatz for interval real number systems can be deduced (see MACINTYRE [4]).

The TARSKI elimination procedure (which is also known as the TARSKI-SEIDENBERG result in algebraic geometry) is a fundamental result in the study of semi-algebraic sets over real closed fields (see [1]). In this context it says that a projection of a semi-algebraic set is again semi-algebraic. The translation (Lemma 4), together with the TARSKI-SEIDENBERG result, should enable a suitable theory of semi-algebraic sets over interval real systems to be developed.

3. Partial Real Closed Fields

As a starting point for the axiomatic characterization of $\mathscr{I}(\mathsf{R})$ given in Theorem 1 we used a set Σ_{R} of axioms obtained by relativizing axioms for real closed fields to the predicate $\boldsymbol{R}x$. While this has a certain logical efficiency it does not produce an aesthetically pleasing set of axioms. The results of this section represent an attempt to replace as many of the axioms in Σ_{R} as possible by natural statements that avoid the $\boldsymbol{R}x$ relativization. To organize the new statements it also seems desirable to isolate the contributions made by the order relation as done in ordinary algebra. For this reason we formulate the concepts of a "partial field", a "partial ordered field", and finally a "partial real closed field". Proceeding in this way it is unreasonable to expect that the underlying partial order $\sqsubseteq$ can be avoided; in fact, it has been used frequently to express properties of the interval reals (see [5], [6]).

How shall we regard structures that are equipped with the "is approximate to" ordering $\sqsubseteq$? The answer is given in SCOTT [8]. They should be viewed as systems of "partial elements", like data types [8], or domains [9], where the elements maximal with respect to $\sqsubseteq$ are regarded as "ideal" elements given with "perfect information". Thus, the goal is to identify (axiomatize) concepts within the "partial algebra" framework which reduced to apropriate ring-theoretic concepts for the "ideal" elements.

A *partial commutative ring (with unity)* is a system

$$\mathscr{S} = \langle S, \sqsubseteq, +, -, \cdot, 0, 1 \rangle$$

that satisfies the axioms:

1. $\sqsubseteq$ is a partial ordering and $+, \cdot, -$ are monotone with respect to $\sqsubseteq$.

2a. $(x + y) + z = x + (y + z)$,

2b. $x + y = y + x$,

2c. $0 + x = x$,

2d. $x + (-x) \sqsubseteq 0$,

2e. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,

2f. $x \cdot y = y \cdot x$,

2g. $x \cdot y + x \cdot z \sqsubseteq x \cdot (y + z)$,

2h. $x \cdot 0 = 0$,

2i. $1 + (-1) = 0$,

2j. $1 \cdot x = 0$,

2k. $(-1) \cdot x = -x$.

A partial commutative ring (with unity) is a *partial field* if

3. $x \neq 0 \rightarrow \exists y \, \exists z (x \sqsubseteq z \,\&\, z \cdot y = 1)$.

The following are easy consequences of the axioms above.

Lemma 5. *The following hold in every partial field* $\mathscr{S}$:

(1) $x = -(-x)$,

(2) $x + y = 0 \rightarrow y = -x$,

(3) $\boldsymbol{R}x \,\&\, x \sqsubseteq y \rightarrow x = y$,

(4) $0 = -0$, *i.e.*, $\boldsymbol{R}0$,

(5) $x + y = 0 \rightarrow \boldsymbol{R}x \,\&\, \boldsymbol{R}y$,

(6) $\boldsymbol{R}x \rightarrow \boldsymbol{R}(-x)$,

(7) $x \cdot y = 1 \rightarrow \boldsymbol{R}x \,\&\, \boldsymbol{R}y$,

(8) $x \neq 0 \,\&\, \boldsymbol{R}x \rightarrow \exists y (\boldsymbol{R}y \,\&\, x \cdot y = 1)$,

(9) $\boldsymbol{R}x \rightarrow x \cdot (y + z) = x \cdot y + x \cdot z$,

(10) $\boldsymbol{R}x \,\&\, \boldsymbol{R}y \rightarrow \boldsymbol{R}(x + y)$,

(11) $\boldsymbol{R}x \,\&\, \boldsymbol{R}y \rightarrow \boldsymbol{R}(x \cdot y)$,

(12) $\boldsymbol{R}x$ *holds iff* x *is maximal with respect to* $\sqsubseteq$.

Proof. (1) Immediate from 2k.

(2) The assumption $x + y = 0$ implies $-x = -x + (x + y) = (-x + x) + y \sqsubseteq 0 + y = y$. Similarly, $-y \sqsubseteq x$ which yields $y \sqsubseteq -x$ since $-$ is monotone with respect to $\sqsubseteq$. Then $y = -x$ follows from axiom 1.

(3) Suppose $x + (-x) = 0$ and $x \sqsubseteq y$. Then $-x \sqsubseteq -y$ by monoticity so

$$0 = x + (-x) \sqsubseteq x + (-y) \sqsubseteq y + (-y) \sqsubseteq 0$$

by 1 and 2d. By 1 and 2b, $-y + x = 0$ so (1) and (2) yield $x = -(-y) = y$.

(4) and (5) are clear from (2) and (6) follows from (1) and 2b.

(7) Suppose $x \cdot y = 1$. Then 2e, 2f and 2k yield $-1 = -(x \cdot y) = x \cdot (-y)$. Thus,

$$0 = 1 + (-1) = x \cdot y + x \cdot (-y) \sqsubseteq x \cdot (y + (-y))$$

by 2i and 2g. Therefore,

$$0 = y \cdot 0 \sqsubseteq y \cdot x \cdot (y + (-y)) = y + (-y)$$

follows from 2h and 1. This yields $\boldsymbol{R}y$. Similarly, $\boldsymbol{R}x$.

(8) Immediate from axiom 3, (3) and (7).

(9) Because of 2h we may assume w.l.o.g. that $x \neq 0$ and $\boldsymbol{R}x$. By (8) there exist u with $u \cdot x = 1$. Now by 2g

$$u \cdot x \cdot y + u \cdot x \cdot z \sqsubseteq u \cdot (x \cdot y + x \cdot z),$$

so $y + z \sqsubseteq u \cdot (x \cdot y + x \cdot z)$. Since $\sqsubseteq$ is monotone,

$$x \cdot (y + z) \sqsubseteq x \cdot u(x \cdot y + x \cdot z) = x \cdot y + x \cdot z$$

from which $x \cdot (y + z) = x \cdot y + x \cdot z$ follows using 2g and 1.

(10) Since $\boldsymbol{R}x$ and $\boldsymbol{R}y$, $0 = x + y + (-x) + (-y) \sqsubseteq (x + y) + (-(x + y))$ using 2b, 2g and 2k. By (3), $x + y + (-(x + y)) = 0$.

(11) Using (9), $x \cdot y + (-(x \cdot y)) = x \cdot y + x \cdot (-y) = x \cdot (y + (-y)) = x \cdot 0 = 0$.

(12) (3) shows that $\boldsymbol{R}x$ implies x is maximal. For the converse, w.l.o.g. we assume $x \neq 0$. Applying axiom 3 and (7), maximality yields $\boldsymbol{R}x$.

The following notation is useful when orderings are discussed. Define

$$x \gtrless y \quad \text{iff} \quad \exists u, v(x \sqsubseteq u \;\&\; u < y \;\&\; y \sqsubseteq v \;\&\; x < v).$$

A system $\mathscr{S} = \langle S, \sqsubseteq, +, -, \cdot, 0, 1, < \rangle$ is a *partial ordered field* if, in addition to the partial field axioms, $\mathscr{S}$ satisfies

4a. $x < y \;\&\; x \sqsubseteq x' \;\&\; y \sqsubseteq y' \rightarrow x' < y'$,

4b. $x < y \;\&\; u < v \rightarrow u < y \vee x < v$,

4c. $x \nless x$,

4d. $x < y \vee y < x \vee x \sqsubseteq y \vee y \sqsubseteq x \vee x \gtrless y \vee y \gtrless x$,

4e. $0 < 1$,

4f. $y < z \rightarrow x + y < x + z \vee x + y \gtrless x + z$,

4g. $y < z \;\&\; 0 < x \rightarrow x \cdot y < x \cdot z \vee x \cdot y \gtrless x \cdot z$.

Lemma 6. *The following hold in every partial ordered field:*

(1) $x < y \,\&\, y < v \rightarrow x < v$,

(2) $(\boldsymbol{R}x \vee \boldsymbol{R}y) \,\&\, x \gtrless y \rightarrow x < y$,

(3) $\boldsymbol{R}x \rightarrow x < y \vee y < x \vee y \sqsubseteq x$,

(4) $\boldsymbol{R}x \,\&\, \boldsymbol{R}y \,\&\, \boldsymbol{R}z \,\&\, y < z \rightarrow x + y < x + z$,

(5) $\boldsymbol{R}x \,\&\, \boldsymbol{R}y \,\&\, \boldsymbol{R}z \,\&\, y < z \,\&\, 0 < x \rightarrow x \cdot y < x \cdot z$.

The proof is left as an exercise.

The following theorem is an easy consequence of Lemmas 5 and 6.

Theorem 4. (1) $\boldsymbol{R}^{(\mathscr{S})}$ *is a field whenever* $\mathscr{S}$ *is a partial field.* (2) $\boldsymbol{R}^{(\mathscr{S})}$ *is a ordered field whenever* $\mathscr{S}$ *is a partial ordered field.*

Finally a partial ordered field $\mathscr{S}$ is called a *partial real closed field* if $\boldsymbol{R}^{(\mathscr{S})}$ satisfies one of the various infinite axiom schemas that makes $\boldsymbol{R}^{(\mathscr{S})}$ a real closed field. For example, we can require the axioms, one for each degree n, that assert: if $f(x)$ is a polynomial of degree n with coefficients in $\boldsymbol{R}^{\mathscr{S}}$, $x_1, x_2 \in \boldsymbol{R}^{\mathscr{S}}$, $x_1 < x_2$ and $f(x_1) < 0 < f(x_2)$, there exist y in $\boldsymbol{R}^{\mathscr{S}}$, $x_1 < y < x_2$ with $f(y) = 0$ (see [3]). It is not clear to the author how to extend the property above to a partial ordered field axiom not involving $\boldsymbol{R}x$.

The set of axioms given above for a partial real closed field can replace the axioms Σ_{R} used to define an interval real field in section 1. Since the language for partial fields includes the symbol $\sqsubseteq$ the following axiom needs to be added to Σ_I.

(I_8) $x \sqsubseteq y$ iff $\underline{x} \leqq \underline{y} \,\&\, \bar{y} \leqq \bar{x}$.

The new set of axioms completely axiomatizes $\mathscr{I}(\mathsf{R})$ in the expanded language.

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