CLONES OF OPERATIONS ON RELATIONS

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It is well known that every Boolean polynomial can be written as a polynomial using only the Sheffer stroke operation $\uparrow$ which is defined by

$$x \uparrow y = \bar{x} + \bar{y}.$$

This fact is expressed by saying that the clone of any Boolean algebra is one-generated (by { $\uparrow$ }). This paper deals with clones of Boolean algebras with additional operations. We show that, for such algebras, finitely generated clones are one-generated, give specific sets of generators for clones of relation algebras, and show that, in at least one case, the generating set given is best possible. Our observations were motivated by questions posed by Bjarni Jonsson in [1]. The original proof of Theorem 2 has been simplified thanks to a suggestion of Roger Maddux.

1. INTRODUCTION.

The first result applies to most of the natural clones which extend the classical Boolean clone.

THEOREM 1. *Every finitely generated clone that contains the Sheffer stroke operation is one-generated.*

In the particular case of relation algebras we show

THEOREM 2. *The clone of every relation algebra is generated by*

(a) *the Sheffer stroke and one binary operation*

$$\tau(x,y) = (x;y^{\smile}) + (1' - (1;(x+y);1)),$$

and

(b) *one ternary operation*

$$\delta(x,y,z) = ((x \uparrow y) - (1;(y \oplus z);1)) + \tau(x,y) \cdot (1;(y \oplus z);1)$$

where $\oplus$ *denotes symmetric difference.*

Is it possible to do better than Theorem 2 ? The following shows that 2(a) cannot be improved.

THEOREM 3. For every set V with at least 6 elements there is a proper relation algebra on V in which relative composition is not contained in the clone generated by the Sheffer stroke operation together with all unary terms of the algebra.

It is not known whether 2(b) can be improved to a single binary operation.

2. THE PROOFS

Proof of Theorem 1. It is shown in [1] that every finitely generated clone that contains $\uparrow$ can be generated by $\{\uparrow,\kappa\}$ for some n-ary term $\kappa(x_1,\ldots,x_n)$. Now, let

$$\rho(y,z,w,x_1,\ldots,x_n) = \bar{y} + \bar{z} + \bar{w}\cdot\kappa(x_1,\ldots,x_n)$$

where $\bar{y}$ denotes $-y$, etc. Then $\rho(y,z,z,x_1,\ldots,x_n) = y\uparrow z$ and $\rho(1,1,\emptyset,x_1,\ldots,x_n) = \kappa(x_1,\ldots,x_n)$. Thus, $\text{clone}(\rho) = \text{clone}(\uparrow,\kappa)$ as desired.

Theorem 2 produces special generating sets for the clone $C_c(\mathfrak{A})$ of all operations definable from the basic operations of the relation algebra $\mathfrak{A} = \langle A,\uparrow,;,{}^\vee,1'\rangle$.

Proof of Theorem 2. To show that the set $\{\uparrow,\tau\}$, given in (a), generates the clone $C_c(\mathfrak{A})$ of a relation algebra $\mathfrak{A}$ it suffices to verify that the identities (i)-(iii) below hold in every relation algebra.

(i) $1' = \tau(\emptyset,\emptyset)$

(ii) $x^\vee = \tau(1',x)$

(iii) $x;y = \tau(x,y^\vee) - \tau(x+y^\vee,\emptyset)$.

It is easily seen that (i) holds in $\mathfrak{A}$. Since $1;(1'+x);1 \geq 1;1';1 = 1$ in every relation algebra $\mathfrak{A}$, $\tau(1',x) = 1';x^\vee + (1'-(1;(1'+x);1)) = x^\vee$ so (ii) holds. To see that (iii) holds in $\mathfrak{A}$ observe that

$$\begin{aligned}\tau(x,y^\vee) - \tau(x+y^\vee,\emptyset) &= [(x;y) + (1'-1;(x+y^\vee);1)] - [1' - 1;(x+y^\vee);1] \\ &= (x;y)\cdot-(1'-1;(x+y^\vee);1) \\ &= (x;y)\cdot(\emptyset' + 1;(x+y^\vee);1) \\ &= x;y\end{aligned}$$

because $x;y \leq 1;(x+y^\vee);1 \leq \emptyset' + 1;(x+y^\vee);1$. It follows that $C_c(\mathfrak{A})$ is generated by $\{\uparrow,\tau\}$.

To show that the operation δ, in (b), generates the clone of a relation algebra $\mathfrak{A}$ it suffices to verify that the identities (iv) and (v) below hold in $\mathfrak{A}$.

(iv) $x \dagger y = \delta(x,y,y)$

(v) $\tau(x,y) = \delta(x,y,-y)$.

Since $y \oplus y = \emptyset$ for every y in $\mathfrak{A}$, $1;(y\oplus y);1 = \emptyset$ from which $\delta(x,y,y) = x\dagger y$ easily follows. So, (iv) holds in $\mathfrak{A}$. Condition (v) follows from the definition of δ because $1;(y\oplus\bar{y});1 = 1;(y+\bar{y});1 = 1;1;1 = 1$ holds in every relation algebra $\mathfrak{A}$. This completes the proof of Theorem 2.

Proof of Theorem 3. The proof will be given in two steps. First, a proper relation algebra on a 6 element set with the desired properties will be constructed and then the general case will be treated.

Let $H = \{ \emptyset,\ldots,5 \}$, $I_H = \{ (i,i) : i \in H \}$, $\mathfrak{A}_\emptyset = \langle Sb({}^2H), \cup, \cap, \sim, \emptyset, {}^2H \rangle$ denote the Boolean algebra of all subsets of ${}^2H$, and let $\mathfrak{A}[H] = \langle \mathfrak{A}_\emptyset, |, {}^{\vee}, I_H \rangle$ denote the (full) proper algebra of relations on H with $\mathfrak{A}_\emptyset$ as its Boolean part. To prove the theorem for $|V| = |H| = 6$, it suffices to find a collection $\mathfrak{B}$ of binary relations on H such that $\mathfrak{B}$ is a Boolean subalgebra of $\mathfrak{A}_\emptyset$, $\mathfrak{B}$ is closed under all unary relation algebra terms, but $\mathfrak{B}$ is not a subuniverse of the relation algebra $\mathfrak{A}[H]$. Consider the following relations on H:

$x = \{ (\emptyset,1),(1,\emptyset),(2,3),(3,2),(4,5),(5,4) \}$

$y = \{ (\emptyset,2),(2,\emptyset),(1,4),(4,1),(3,5),(5,3) \}$

$z = \{ (\emptyset,5),(5,\emptyset),(1,3),(3,1),(2,4),(4,2) \}$

$u = \{ (\emptyset,4),(1,2),(2,5),(3,\emptyset),(4,3),(5,1) \}$

$v = \{ (\emptyset,3),(1,5),(2,1),(3,4),(4,\emptyset),(5,2) \}$

$w = u \cup v$

Let $\mathfrak{B}$ denote the Boolean subalgebra of $\mathfrak{A}_\emptyset$ generated by $\mathcal{P} = \{ I,x,y,z,w \}$. Since $\mathcal{P}$ partitions ${}^2H$, an elements of $\mathfrak{B}$ is just a union of a set of generators. To see that $\mathfrak{B}$ is the desired model it suffices to show

(i). $x|y \notin B$, and

(ii). for every $a \in \mathfrak{B}$ the subalgebra of $\mathfrak{A}[H]$ generated by a, $\mathfrak{Sg}(a)$, is contained in $\mathfrak{B}$.

Note that property (ii) implies that $\mathfrak{B}$ is closed under every unary relational term. To assist in the verification of (i) and (ii), it is convenient to know how the relative composition (partial) operation $|$ works on the atoms of $\mathfrak{B}$. The "multiplication" table for $|$ is given below in Table 1.

From Table 1, $x|y = u$; so (i) holds since $u \notin \mathfrak{B}$. For $a \in \mathfrak{B}$ observe that $\mathfrak{Sg}\{ a \} = \mathfrak{Sg}\{ \sim a \}$ and $\mathfrak{Sg}\{ a \} = \mathfrak{Sg}\{ a \sim I \}$. Hence it suffices to verify the conclusion of (ii) for elements a which are the union of at most two atoms in the set $\{ x,y,z,w \}$. For $a = \emptyset$, $\mathfrak{Sg}\{ a \}$ is the minimal subalgebra of $\mathfrak{A}[H]$ which is contained in $\mathfrak{B}$. If $a \in \{ x,y,z,w \}$, let $\hat{a}$ denote $\sim a \cap \sim I$. It is easily seen that the atoms of $\mathfrak{Sg}\{ a \}$ are I, a, and $\hat{a}$ because $a;\hat{a} = \hat{a} = \hat{a};a$ and $\hat{a};\hat{a} = 1$. Hence $\mathfrak{Sg}\{ a \} \subseteq \mathfrak{B}$ when a is an atom of $\mathfrak{B}$. Now, suppose that a is the union of two atoms in the set $\{x,y,z,w\}$. As before let $\hat{a} = {\sim}a \cap {\sim}I$. Either $a \geq w$ or $\hat{a} \geq w$. Without loss of generality we assume that $\hat{a} \geq w$ and set $a' = \hat{a} \sim w$. The atoms of $\mathfrak{Sg}\{a\}$ are I, a, a' and w as can be seen from the "multiplication" table of $|$ for these relations (Table 2).

	I	x	y	z	w
I	I	x	y	z	w
x	x	I	u	v	$y \cup z$
y	y	v	I	u	$x \cup z$
z	z	u	v	I	$x \cup y$
w	w	$y \cup z$	$x \cup z$	$x \cup y$	$I \cup w$

Table 1

	I	a	a'	w
I	I	a	a'	w
a	a	$I \cup w$	w	$a \cup a'$
a'	a'	w	I	a
w	w	$a \cup a'$	a	$I \cup w$

Table 2

It follows that $\mathfrak{Sg}\{a\} \subseteq \mathfrak{K}$ when a is a join of two atoms of $\mathfrak{K}$. Hence (ii) holds which completes the proof of Theorem 3 when $|V| = 6$.

Now suppose $|V| \geq 6$ and $H = \{\emptyset,\ldots,5\} \subseteq V$. Let $G = V{\sim}H$ and $N = \{I_G, {}^2G{\sim}I_G, G{\times}H, H{\times}G\}$. Then, recalling notation from the first case, $\mathcal{P} \cup N$ partitions ${}^2V$ and, for every $a,b \in N$, $\{a^{\smile}, a;b\} \subseteq N \cup \{\emptyset, {}^2G\}$. Let $\mathfrak{K}'$ be the Boolean algebra generated by $\mathcal{P} \cup N$. As in the $|V| = 6$ case, it suffices to show that (i) and (ii) hold for the modified $\mathfrak{K}'$.

It is clear that $x;y \notin \mathfrak{K}'$, because, for example, every element of $\mathfrak{K}$ is symmetric. Thus, (i) holds for $\mathfrak{K}'$.

To show (ii) for $\mathfrak{K}'$ we shall use the fact that (ii) holds for $\mathfrak{K}$ and that the domain and range of every $b \in \mathfrak{K}$ is H. Suppose $a \in \mathfrak{K}'$. Then $a = b + \Sigma X$ for some $b \in \mathfrak{K}$ and some $X \subseteq N$. Let $\mathfrak{S}$ be the relation subalgebra of $\mathfrak{R}[H]$ generated by $\{b\}$. Then $\mathfrak{S} \subseteq \mathfrak{K}$ since (ii) holds for $\mathfrak{K}$. Let $\mathfrak{S}'$ be the Boolean algebra generated by $\mathfrak{S} \cup N$. Since $a \in \mathfrak{S}' \subseteq \mathfrak{L}'$, in order to establish (ii) for $\mathfrak{L}'$, it is enough to show that $\mathfrak{S}'$ contains I_V and is closed under $^\vee$ and $;$. It is clear that $I_V = I_H \cup I_G$ belongs to $\mathfrak{S}'$. Now, suppose $a \in \mathfrak{S}'$ has the form $a = b + \Sigma X$ for some $b \in \mathfrak{S}$ and some $X \subseteq N$. Then $a^\vee = b^\vee + \Sigma\{ t^\vee : t \in X \}$ belongs to $\mathfrak{S}'$ since $b^\vee \in \mathfrak{S}$ and N is closed under $^\vee$. Hence $\mathfrak{S}'$ is closed under $^\vee$. Now, suppose $a,c \in \mathfrak{S}'$ where $a = b + \Sigma X$ and $c = d + \Sigma Y$ with $b,d \in \mathfrak{S}$ and $X,Y \subseteq N$. Then

$$a;c = b;d + b;\Sigma Y + \Sigma X;d + \Sigma X;\Sigma Y.$$

Since $b;d \in \mathfrak{S}$, $b;\Sigma Y \in \{ \emptyset, H \times G \}$, $\Sigma X;d \in \{ \emptyset, G \times H \}$, and $\Sigma X;\Sigma Y \in N \cup \{ \emptyset, {}^2G \}$, it follows that $a;c \in \mathfrak{S}'$. Thus, $\mathfrak{S}'$ is a relation subalgebra of $\mathfrak{A}[V]$ and it follows that (ii) holds for $\mathfrak{L}'$ which completes the proof of Theorem 3.

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