Convegno su:
IPERGRUPPI, ALTRE STRUTTURE MULTIVOCHE E LORO APPLICAZIONI
Editor: P. CORSINI
UDINE, 15 - 18 Ottobre 1985

THE CAYLEY REPRESENTATION OF POLYGROUPS

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It is well known that every ordinary group can be represented in a concrete way as a group of permutations. In this paper we consider polygroups, special multivalued group-like systems, and investigate a multivalued analogue of the classical group result; namely, representation by polygroups of generalized permutations. We call a polygroup Cayley if it satisfies a natural multivalued version of the Cayley representation result. Many natural polygroups are Cayley although there exist non-Cayley systems. The main result is a characterization of Cayley polygroups in terms of the existence of graph colorings.

1. Generalized permutations and Cayley polygroups

For a set X, the set of all non-empty subsets of X is denoted by $S_\emptyset(X)$. A <u>polygroup</u> $\mathcal{G}$ is a completely regular, reversible-in-itself multigroup in the sense of Dresher and Ore [4]. More precisely, $\mathcal{G}$ is a system $\langle G, \cdot, ^{-1}, e\rangle$ where $e \in G$, $^{-1}: G \to G$, $\cdot : G^2 \to S_\emptyset(G)$ and the following axioms hold for all $x,y,z \in G$:

(i) $(x\cdot y)\cdot z = x\cdot(y\cdot z)$

(ii) $x\cdot e = x = e\cdot x$

(iii) $x \in y\cdot z$ implies $y \in x\cdot z^{-1}$ and $z \in y^{-1}\cdot x$.

Commutative polygroups are the same as the canonical hypergroupes studied by Mittas [6] and this notion has been shown to be equivalent to the notion of a

[1]Research supported in part by a grant from The Citadel Development Foundation.

join space with identity in Corsini [3].

A <u>generalized permutation</u> on a set X is a function $f:X \to S_\emptyset(X)$ such that $X = \bigcup \{f(x) : x \in X\}$. The collection of all generalized permutations on X is denoted by GPrm(X). A generalized permutation on a finite set X corresponds to a digraph with vertex set X for which the indegree and outdegree of each vertex is at least 1.

Various results on permutations extend to generalized permutations via graph theory. For example, call $f \in \mathrm{GPrm}(X)$ a <u>generalized cycle</u> if, for every $x,y \in X$, there is a sequence $x = x_\emptyset, x_1, \ldots, x_k = y$ such that $x_{j+1} \in f(x_j)$ or $x_j \in f(x_{j+1})$ for all $j < k$. It can be assumed without loss of generality that all the x_i's are distinct. The following result is essentially the graph-theoretic result that every digraph has a unique decomposition into weakly connected components.

1. <u>PROPOSITION</u>. *Every generalized permutation on a finite set can be written uniquely (up to the order of factors) as a union of disjoint generalized cycles.*

In order to form a polygroup of generalized permutations we must treat the composition of these functions. For $f,g \in \mathrm{GPrm}(X)$ define $fg : X \to S_\emptyset(X)$ by

$$(fg)(x) = g(f(x)) = \cup\{g(y) : y \in f(x)\}.$$

A set $F \subseteq \mathrm{GPrm}(X)$ is <u>closed under composition</u> if for every $f,g \in F$ there exist $K(f,g) \subseteq F$ such that for every $x \in X$

$$(fg)(x) = \bigcup \{h(x) : h \in K(f,g)\}.$$

In this case the <u>product</u> $f*g$ of $f,g \in F$ is $f*g = K(f,g)$. The inverse of $f \in \mathrm{GPrm}(X)$ is defined by $f^{-1}(x) = \{y \in X : x \in f(y)\}$. It is clear that

f^{-1} is again a generalized permutation.

A collection $F \subseteq GPrm(X)$ is called <u>regular</u> if for every $x, y \in X$, $y \in f(x)$ for some $f \in F$.

For a set $F \subseteq GPrm(X)$ that is closed under composition, closed under the inverse operation, and contains $I_X = \{(x,x) : x \in X\}$, it can be proved that the system $\mathcal{F} = \langle F, *, ^{-1}, I_X \rangle$ is a polygroup. We call $\mathcal{F}$ a <u>functional polygroup</u>. If F is regular, we say that $\mathcal{F}$ is a <u>regular functional polygroup</u>.

For a polygroup $\mathcal{G}$ and $F \subseteq GPrm(X)$, a function $\sigma : G \to F$ is <u>faithful</u> if, for all $a, b \in G$, and $x \in X$, $\sigma(a)(x) \cap \sigma(b)(x) = \emptyset$ whenever $a \neq b$. A polygroup $\mathcal{G}$ is <u>Cayley</u> if there is a faithful isomorphism of $\mathcal{G}$ onto a regular functional polygroup.

Does the analogue of Cayley's theorem hold? Almost!

2. <u>THEOREM</u>. *Every polygroup is isomorphic to a regular functional polygroup. (In general, the isomorphism is <u>not</u> faithful.)*

Proof. Given a polygroup $\mathcal{G} = \langle G, \cdot, ^{-1}, e \rangle$, define, for each $a \in G$, $f_a \in GPrm(G)$ by $f_a(x) = x \cdot a$ (for all $x \in G$). It is routine to check that $F = \{f_a : a \in G\}$ is closed under composition with $K(f_a, f_b) = \{f_c : c \in a \cdot b\}$, $f_a^{-1} = f_{a^{-1}}$, and $I_G = f_e$. Moreover, it is clear that F is regular: namely, given $a, b \in G$, $b \in f_x(a)$ for every solution x to $b \in a \cdot x$. Thus, $\mathcal{F} = \langle F, *, ^{-1}, I_G \rangle$ is a regular functional polygroup. The map $\sigma : G \to F$ given by $\sigma(a) = f_a$ preserves the operations and is one-one, so $\mathcal{G} \cong \mathcal{F}$. Examples given below will show that, in general, σ is not faithful. □

3. <u>REMARKS</u>. (1). Not all polygroups are Cayley. For example the 4 element system $\mathcal{A}$ with multiplication table below is not Cayley. See Proposition 6 in section 2.

	Ø	1	2	3
Ø	Ø	1	2	3
1	1	13	Ø123	12
2	2	Ø123	23	12
3	3	12	12	Ø3

(2). By Theorem 5 in section 2 and the results in section 3 of [1] the problem of deciding whether or not a polygroup is Cayley is equivalent to deciding whether an integral relation algebra is representable. Using this connection we can restate a result of Lyndon about projective geometry. In [7] Prenowitz associated a join space with identity (a commutative polygroup) $\mathcal{P}_G$ to every projective geometry G. In the context of relation algebras Lyndon proved in [5] that a projective geometry G is embeddable as a hyperplane in a geometry of one higher dimension if and only if $\mathcal{P}_G$ is Cayley. Consequently, for each n such that there is no projective plane with n points on a line, there is an associated non-Cayley polygroup. See [2] for details.

2. An alternate characterization of Cayley polygroups.

In this section we characterize to notion of a Cayley polygroup in terms of the existence of colorings of a complete graph.

Let C be a set with $|C| \geq 2$ and $\emptyset \in C$. (Think of C as a set of colors and $\emptyset$ as the neutral color.) Also, suppose $\iota : C \to C$ such that

$\iota^2 = I_{\mathcal{C}}$ and $\iota(\emptyset) = \emptyset$. A color scheme (of type $\mathcal{C}$) is a system $\mathcal{V} = \langle V, \{C_a : a \in \mathcal{C}\}\rangle$ such that $C_a \subseteq V^2$ for all $a \in \mathcal{C}$ and the following properties hold: ($^\vee$ denotes relation converse and $|$ denotes composition.)

(CS1) $\{C_a : a \in \mathcal{C}\}$ partitions V^2 and $C_\emptyset = I_V$,

(CS2) $(\forall a\in\mathcal{C})(\ C_{\iota(a)} = C_a^\vee\ )$,

(CS3) $(\forall a\in\mathcal{C})(\forall x\in V)(\exists y\in V)(\ (x,y) \in C_a\ )$,

(CS4) $(\forall a,b,c\in\mathcal{C})(\ C_c \cap (C_a | C_b) \neq \emptyset$ implies $C_c \subseteq C_a | C_b\ )$,

ie., for $a,b,c \in \mathcal{C}$ and $x,y \in V$ $\{z\in V : xC_a z \ \&\ zC_b y\} \neq \emptyset$ is independent of $(x,y) \in C_c$.

The following lemma is proved in a straightforward way.

4. LEMMA. *For a color scheme* $\mathcal{V} = \langle V, \{C_a : a \in \mathcal{C}\}\rangle$ *the system* $\mathfrak{M}_{\mathcal{V}} = \langle \mathcal{C}, \cdot, \iota, \emptyset\rangle$ *where* $a \cdot b = \{c \in \mathcal{C} : C_c \subseteq C_a | C_b\}$ *is a polygroup.*

The system $\mathfrak{M}_{\mathcal{V}}$ defined in the lemma is called the color algebra of the color scheme $\mathcal{V}$. We say that a polygroup $\mathcal{G}$ is chromatic if $\mathcal{G} \cong \mathfrak{M}_{\mathcal{V}}$ for some color scheme $\mathcal{V}$. The next theorem is the main characterization.

5. THEOREM. *A polygroup is Cayley if and only if it is chromatic.*

Proof. ($\Rightarrow$) Suppose σ is a faithful isomorphism of $\mathcal{G}$ onto a regular functional polygroup $\mathcal{F} \subseteq \mathrm{GPrm}(X)$. We build a color scheme on X as follows: let $\mathcal{C} = \mathcal{G}$, $\emptyset = e$, $\iota(a) = a^{-1}$, and, for $a \in \mathcal{C}$, $C_a = \{(x,y) \in X^2 : y \in \sigma(a)(x)\}$. Then $\mathcal{V} = \langle X, \{C_a : a\in\mathcal{C}\}\rangle$ is a color scheme and $\mathcal{G} = \mathfrak{M}_{\mathcal{V}}$.
($\Leftarrow$) It is enough to show that $\mathfrak{M}_{\mathcal{V}}$ is Cayley where $\mathcal{V} = \langle V, \{C_a : a\in\mathcal{C}\}\rangle$ is a color scheme. For $a \in \mathcal{C}$ define $f_a(x) = \{y \in V : (x,y) \in C_a\}$ for all $x \in V$. Then, by (CS3) $f_a \in \mathrm{GPrm}(V)$, by (CS1) $f_\emptyset = C_\emptyset = I_V$, by (CS4) $f_a f_b = \{f_c : c \in a\cdot b \text{ (in } \mathfrak{M}_{\mathcal{V}})\}$, and by (CS2) $(f_a)^{-1} = f_{a^{-1}}$. Moreover, by

(CS1) $F = \{f_a : a \in C\}$ is regular; thus, $\mathcal{F} = \langle F, *, ^{-1}, I_V\rangle$ is a regular functional polygroup. The isomorphism $\sigma: a \to f_a$ of $\mathcal{M}_V$ onto $\mathcal{F}$ is faithful by (CS1) because $z \in f_a(x) \cap f_b(x)$ if and only if $(x,z) \in C_a \cap C_b$. □

The characterization in Theorem 5 provides a graphical approach for determining if a polygroup is Cayley. We give two results below to illustrate its use.

6. <u>PROPOSITION</u>. *The polygroup $\mathcal{A}$ in Remark 3 is not Cayley.*

Proof. Suppose $\mathcal{A}$ is the color algebra of a scheme $\mathcal{V} = \langle V, C_\emptyset, C_1, C_2, C_3\rangle$ where ι interchanges 1 and 2 and fixes 3. The multiplication table for $\mathcal{A}$ tells which colored triangles in $\mathcal{V}$ must be present and which are forbidden. By (CS3) there exist $(x,y)\in C_3$. Since $3\in 1\cdot 1$ and $3\in 2\cdot 2$, so there exist u and v with $(x,u)\in C_1$, $(u,y)\in C_1$, $(x,v)\in C_2$, and $(v,y)\in C_2$. Now, $(u,v)\in(C_2|C_2)\cap(C_1|C_1) = C_3$ from the multiplication table [remember $\iota(1) = 2$]. Since $1\in 1\cdot 1$ there exist w with $(y,w)\in C_1$ and $(w,v)\in C_1$. From the multiplication table for $\mathcal{A}$ we can determine the colors on the edges (w,x) and (w,u): $(w,x)\in(C_2|C_3)\cap(C_1|C_1) = C_1$ and $(w,u)\in(C_2|C_2)\cap(C_1|C_3) = C_2$. Because the triangle $\{w,x,u\}$ has $(w,x)\in C_1$, $(x,u)\in C_1$ and $(w,u)\in C_2$ it follows that the $2\in 1\cdot 1$ holds in $\mathcal{A}$ which is a contradiction. □

As a second illustration of the use of Theorem 5 we show that many natural polygroups are Cayley. For a group $\mathcal{G}$, an equivalence relation θ on G is called a (full) <u>conjugation</u> if (i) $\theta(xy) \subseteq (\theta x)(\theta y)$ and (ii) $(\theta x)^{-1} = \theta(x^{-1})$ for all $x,y \in G$. For a conjugation θ on a group $\mathcal{G}$ a quotient polygroup $\mathcal{G}//\theta$ can be defined on the θ-blocks in a natural way. Examples of full conjugations can be found in [1]. From Theorem 2.3 of [1] we immediately obtain

7. PROPOSITION. *For every group $\mathcal{G}$ and full conjugation θ on $\mathcal{G}$, $\mathcal{G}//\theta$ is Cayley.*

3. Problems.

(1). Is every Cayley polygroup isomorphic to a quotient $\mathcal{G}//\theta$ where θ is a full conjugation on some group $\mathcal{G}$?

(2). Are either of the two polygroups below Cayley?

	Ø	1	2	3
Ø	Ø	1	2	3
1	1	12	Ø123	13
2	2	Ø123	12	23
3	3	13	23	Ø123

	Ø	1	2	3
Ø	Ø	1	2	3
1	1	13	Ø123	123
2	2	Ø123	23	123
3	3	123	123	Ø12

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