Algebra Universalis, 25 (1988) 131–146 0002-5240/88/020131-16$01.50 + 0.20/0

An abstract theory of invertible relations

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The purpose of this paper is to present certain results arising from a study of quasi-orderings (pre-orderings). We show that to each relation $R \subseteq X \times Y$ there are associated unique largest quasi-orderings $\pi_l(R)$ on X and $\pi_r(R)$ on Y such that $\pi_l(R) \circ R \circ \pi_r(R) = R$; and we present formulas for these quasi-orderings. For a fixed pair of quasi-orders π_1 and π_2 we characterize the invertible relations (with respect to the units π_1 and π_2) in terms of isomorphisms between $^*\pi_1$ and $^*\pi_2$, where $^*\pi_i$ is the partial ordering naturally induced by π_i. In particular we show that the set of invertible relations with $\pi_1 = \pi_2 = \pi$ is a group isomorphic to the group Aut $(^*\pi)$ of automorphisms of $^*\pi$. We present these results in sections 1–3 in the framework of a general relation algebra.

In section 4 we describe an anti-isomorphism between the lattice of quasi-orderings on a set X and a certain lattice of topologies on X. Using this anti-isomorphism, we obtain a characterization of the set of relations $R \subseteq X \times Y$ with fixed left and right units π_1 and π_2.

1. Quasi-ordered elements in a relation algebra

The notion of a relation algebra can be defined in several ways. We prefer the definition given in Jónsson–Tarski [4] (Def. 4.1) augmented by the inclusion of complementation as a fundamental operation. A *relation algebra* (a RA for short) is an algebra $\mathfrak{A} = \langle \mathfrak{A}_0, ;, 1', ^{\cup} \rangle$ where

(1) $\mathfrak{A}_0 = \langle A, +, 0, \cdot, 1, ^- \rangle$ is a Boolean algebra,
(2) $x;(y;z) = (x;y);z$ for all $x, y, z \in A$,

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[1] The work of both authors was partly supported by grants from The Citadel Development Foundation.

MR Subject Classification 03G15, 06A99.

Presented by Bjarni Jónsson. Received January 12, 1987. Accepted for publication in final form August 22, 1987.

(3) $1' ; x = x = x ; 1'$ for all $x \in A$,
(4) the conditions $(x ; y) \cdot z = 0$, $(x^{\cup} ; z) \cdot y = 0$, and $(z ; y^{\cup}) \cdot x = 0$ are equivalent for all $x, y, z \in A$.

A standard example of a relation algebra is the algebra $\mathcal{R}e(X) = \langle Sb(X^2), \circ, I_X, ^{-1} \rangle$ of all binary relations on a set X. $Sb(X^2)$ is the Boolean algebra of all subsets of X^2, I_X is the identity relation on X, and, for all $R, S \subseteq X^2$, the composition operation $\circ$ and the inverse operation $^{-1}$ are defined by

$$R \circ S = \{(x, y) : \exists x (xRz \text{ and } zSy)\}$$
$$R^{-1} = \{(x, y) : (y, x) \in R\}.$$

The operations $\circ$, $^{-1}$ and the element I_X correspond to the symbols ;, $^{\cup}$ and $1'$ in the relation algebra definition. More information on relation algebras can be obtained from Chin–Tarski [2], Jónsson [3], or Jónsson–Tarski [4]. Most of the arithmetic properties used below are immediate from the axioms. The following property will be used in the proof of 3.5.

LEMMA 1.1 (Chin–Tarski [2], 2.7). $(x ; y) \cdot z \leq x ; ((x^{\cup} ; z) \cdot y)$.

We call an element x in a relation algebra $\mathfrak{A}$ an *equivalence element* if $x ; x \leq x$, $x^{\cup} \leq x$ and $1' \leq x$. This notion is stronger than the notion defined in [2], [3], and [4] because we require $1' \leq x$.

DEFINITION 1.2. If e is an equivalence element in a RA $\mathfrak{A}$, an element a in $\mathfrak{A}$ is called

(i) a *quasi-order element with respect to* e if $a ; a \leq a$ and $e \leq a$.
(ii) a *partial order element with respect to* e if $a ; a \leq a$, $e \leq a$, and $a \cdot a^{\cup} \leq e$.

Note that if a is a quasi-order (partial order) element with respect to e, then $a ; a = a$. A quasi-order (partial order) element with respect to $1'$ is called a *quasi-order* (*partial order*) *element* in $\mathfrak{A}$.

An equivalence element e in a RA $\mathfrak{A}$ gives rise to another RA called a *factor algebra* (see [3]) that is denoted as $e ; \mathfrak{A} ; e$. The universe of the factor algebra is

$$e ; A ; e = \{e ; a ; e : a \in A\} = \{x \in A : x = e ; x ; e\}.$$

The operations ;, $^{\cup}$, $+$, and $\cdot$ are the same as in $\mathfrak{A}$, the unit is $e ; 1 ; e$, the identity is e, and the complement of x is $x^{-} \cdot (e ; 1 ; e)$. An equivalence element E in $\mathcal{R}e(X)$ is an equivalence relation on X. The factor algebra $E ; \mathcal{R}e(X) ; E$ is naturally isomorphic to $\mathcal{R}e(X/E)$ where X/E is the set of E-blocks.

Because $a;a \le a$ and $e \le a$ imply $a = e;a;e$, the quasi-order (partial order) elements with respect to e in a RA $\mathfrak{A}$ are exactly the quasi-order (partial order) elements in $e;\mathfrak{A};e$. Hence a quasi-order (partial order) element q with respect to an equivalence element E in $\mathscr{R}e(X)$ is exactly a quasi-ordered relation (partially ordered relation) on the set X/E.

The following result is a relation algebra version of the construction of a partial ordering from a quasi-ordering.

LEMMA 1.3. *If q is a quasi-order element in a RA $\mathfrak{A}$ and $e = q \cdot q^{\cup}$, then e is an equivalence element and q is a partial order element with respect to e, i.e., q is a partial order element in $e;\mathfrak{A};e$.*

DEFINITION 1.4. For an element a in a *RA* $\mathfrak{A}$.

(i) the element $\pi_l(a) = (a^-;a^{\cup})^-$ is called the *left unit* of a.

(ii) the element $\pi_r(a) = (a^{\cup};a^-)^-$ is called the *right unit* of a.

If the operation † is defined by $a^\dagger = a^{\cup -}$ (cf., [2], p. 348), the formulas in 1.4 are equivalent to

$$\pi_l(a) = (a;a^\dagger)^\dagger \quad \text{and} \quad \pi_r(a) = (a^\dagger;a)^\dagger.$$

It is easily seen that

$$\begin{aligned} \pi_l(a^-) &= \pi_l(a)^{\cup}, & \pi_r(a^-) &= \pi_r(a)^{\cup}, \\ \pi_l(a^{\cup}) &= \pi_r(a)^{\cup}, & \pi_r(a^{\cup}) &= \pi_l(a)^{\cup}, \\ \pi_l(a^\dagger) &= \pi_r(a) \quad \text{and} & \pi_r(a^\dagger) &= \pi_l(a). \end{aligned}$$

The next lemma shows that the left unit $\pi_l(a)$ (right unit $\pi_r(a)$) is the unique left (right) residual of a over a in the sense of Birkhoff [1].

LEMMA 1.5. (i) *$\pi_l(a)$ is the unique largest solution x to $x;a = a$.*

(ii) *$\pi_r(a)$ is the unique largest solution x to $a;x = a$.*

Proof. (i) Using (4) in the RA definition one obtains

(5) $x;a \le a$ iff $x \le (a^-;a^{\cup})^-$.

Thus, for any solution x to $x;a = a$, $x \le \pi_l(a)$. On the other hand, (5) gives $1' \le (a^-;a^{\cup})^-$ and $a = 1';a \le (a^-;a^{\cup})^-;a \le a$ so $\pi_l(a)$ is a solution to $x;a = a$. The proof of (ii) is similar. □

The next result shows that the units are quasi-order elements and this, in turn, leads to a characterization of quasi-order elements.

LEMMA 1.6. *For every element a in a* RA $\pi_l(a)$ *and* $\pi_r(a)$ *are quasi-order elements.*

Proof. Lemma 1.5(i) implies $1' \leq \pi_l(a)$ and $\pi_l(a);a = a$. Abbreviating $\pi_l(a)$ as π, it follows that

$$a^- \cdot (\pi;\pi;a) = a^- \cdot (\pi;a) = a^- \cdot a = 0$$

because $\pi;a = a$. Hence $(\pi;\pi) \cdot (a^-;a^\cup) = 0$ and thus $\pi;\pi \leq \pi$ using (4). The proof that $\pi_r(a)$ is a quasi-order element is similar. □

COROLLARY 1.7. *The following are equivalent for each a in a* RA:
(i) *a is a quasi-order element,*
(ii) $a = \pi_l(a)$.
(iii) $a = \pi_r(a)$.

Proof. By 1.6, (ii)$\Rightarrow$(i). Now, assume (i). Then $a;a \leq a$ is equivalent to $a \leq \pi_l(a)$. Also, $1' \leq a$ implies $1' \leq a^\cup$, so $a^- = a^-;1' \leq a^-;a^\cup$ which yields $\pi_l(a) = (a^-;a^\cup)^- \leq a^{--} = a$. The proof that (i)$\Leftrightarrow$(iii) is similar. □

2. Invertible relations

Let $Q(\mathfrak{A})$ denote the collection of all quasi-order elements in a complete RA $\mathfrak{A}$. Note that $Q(\mathfrak{A})$ is closed under arbitrary meets so $Q(\mathfrak{A})$ forms a complete lattice which is a meet-sublattice of $\mathfrak{A}$. It is also clear that the map $x \mapsto x^\cup$ is an involution of $Q(\mathfrak{A})$. We extend the operations $^-$, $^\cup$, and † of a relation algebra $\mathfrak{A}$ to subsets of $\mathfrak{A}$ in the obvious way; for example, $X^- = \{x^- : x \in X\}$ whenever X is a subset of $\mathfrak{A}$. For $q_1, q_2 \in Q(\mathfrak{A})$, let

$$R(q_1, q_2) = \{a \in A : \pi_l(a) = q_1, \pi_r(a) = q_2\}.$$

With this notation, the identities following 1.4 show that $R(q_1, q_2)^- = R(q_1^\cup, q_2^\cup)$, $R(q_1, q_2)^\cup = R(q_2^\cup, q_1^\cup)$ and $R(q_1, q_2)^\dagger = R(q_2, q_1)$.

DEFINITION 2.1. (i).The *quasi-inverse* of an element a in a RA $\mathfrak{A}$ is the element $a^\sim = (a^\cup;a^-;a^\cup)^-$.

(ii) An element $a \in R(q_1, q_2)$ is *invertible* if there exist $b \in R(q_2, q_1)$ such that $a;b = q_1$ and $b;a = q_2$. We call b an *inverse* of a.

In terms of the † operation, $a^\smile = (a;a^\dagger;a)^\dagger$.

LEMMA 2.2. *For an element* $a \in R(q_1, q_2)$
(i) $a^\smile$ *is the largest* x *such that* $a;x \leq q_1$.
(ii) $a^\smile$ *is the largest* x *such that* $x;a \leq q_2$.
(iii) If a is invertible, the inverse is unique and equal to $a^\smile$.

Proof. (i) From 1.7, 1.4(i), and (4),

$$a;x \leq q_1 \quad \text{iff} \quad (a;x)\cdot(a^-;a^\cup) = 0 \quad \text{iff} \quad x\cdot(a^\cup;a^-;a^\cup) = 0$$
$$\text{iff} \quad x \leq a^\smile.$$

(ii) Similar to (i).
(iii) Suppose b is an inverse of a. Then $b \leq a^\smile$ by (i). On the other hand,

$$a^\smile = 1';a^\smile \leq q_2;a^\smile = b;a;a^\smile \leq b;q_1 = b,$$

so $a^\smile = b$. □

For $q, q_1, q_2 \in Q(\mathfrak{A})$ let

$G^{\mathfrak{A}}(q_1, q_2) = \{a \in R(q_1, q_2) : a \text{ is invertible}\}$

and $G^{\mathfrak{A}}(q) = G^{\mathfrak{A}}(q, q)$. We write $G(q_1, q_2)$ and $G(q)$ if the RA $\mathfrak{A}$ is understood.
We conclude this section with some observations on $G(q)$.

LEMMA 2.3. *For* $q \in Q(\mathfrak{A})$, $G(q)$ *is a group under* ;

Proof. Clearly, q is the identity element of $G(q)$ and each element is invertible so it suffices to assume $a,b \in G(q)$ and show that $a;b \in G(q)$. Suppose $\pi_l(a;b) = p$. Then $q \leq p$ because $q = \pi_l(a)$ gives $q;a;b = a;b$. Since b is invertible,

$$p;a = p;a;q = p;a;b;b^\smile = a;b;b^\smile = a;q = a.$$

Therefore, $p \leq q$ and $q = \pi_l(a;b)$. Similarly, $\pi_r(a;b) = q$. It is easy to see that $a;b$ is invertible so $a;b \in G(q)$. □

LEMMA 2.4. *If* $q \in Q(\mathfrak{A})$, $e = q \cdot q^{\cup}$ *and* $\mathfrak{B} = e;\mathfrak{A};e$, *then*
(i) $R^{\mathfrak{A}}(q, q) \subseteq e;A;e$
(ii) $G^{\mathfrak{A}}(q) = G^{\mathfrak{B}}(q)$.

Proof. (i). Since $q = e;q = q;e$ and $q;a;q = a$ for $a \in R(q, q)$,

$$a = e;q;a;q;e = e;a;e \in e;A;e.$$

(ii) First note that for $a \in R(q, q)$, $a \leq e;1;e$ by 2.5(i) so the complement of a is the same in both $\mathfrak{A}$ and $\mathfrak{B}$. It follows that $G^{\mathfrak{A}}(q) = G^{\mathfrak{B}}(q)$ because $\pi_l(a)$, $\pi_r(a)$, and $a^{\sim}$ are the same in $\mathfrak{A}$ and $\mathfrak{B}$. □

3. A characterization of $G(q_1, q_2)$.

In this section an invertible element in a RA is characterized by abstracting the idea of a bijection between sets. (See 3.10.)

DEFINITION 3.1. Suppose $\mathfrak{A}$ is a RA, e_1 and e_2 are equivalence elements in $\mathfrak{A}$, and q_1, q_2 are quasi-order elements with respect to e_1, e_2 respectively. Then

(i) $f \in A$ is a *bijection element from* e_1 *to* e_2 if $f^{\cup};f = e_2$, $f;f^{\cup} = e_1$, $e_1;f = f$ and $f;e_2 = f$.
(ii) $f \in A$ is an *isomorphism element of* q_1 *onto* q_2 if f is a bijection element from e_1 to e_2 and $f^{\cup};q_1;f = q_2$.

We denote the collection of all isomorphism elements of q_1 onto q_2 by $\text{Ism}(q_1, e_1 : q_2, e_2)$. Observe that $\text{Ism}(q_1, e_1 : q_2, e_2) \subseteq R(e_1, e_2)$ and if $f \in \text{Ism}(q_1, e_1 : q_2, e_2)$, then $f^{\cup} \in \text{Ism}(q_2, e_2 : q_1, e_1)$. Let $\text{Aut}(q_1, e_1) = \text{Ism}(q_1, e_1 : q_1, e_1)$, $\text{Aut}(q_1) = \text{Aut}(q_1, 1')$ and call the elements of these sets *automorphisms* of q_1.

The lemma below is a routine calculation using 1.2 and 3.1.

LEMMA 3.2. $\text{Aut}(q, e)$ *is a group under* ; *whenever* q *is a quasi-order element with respect to e. The (group) inverse of* $f \in \text{Aut}(q, e)$ is $f^{\cup}$.

For $f \in \text{Ism}(q_1, e_1 : q_2, e_2)$ where q is a quasi-order element with respect to e_i $(i = 1, 2)$, we define $f^* = q_1;f;q_2$. Observe that f^* depends not only on f but also on the quasi-order elements q_1 and q_2. This notation will not cause a problem since the appropriate quasi-order elements will be clear from the context.

In the lemmas below properties of isomorphisms are developed using the arithmetic of relation algebras.

LEMMA 3.3. *For* $f \in \mathrm{Ism}(q_1, e_1 : q_2, e_2)$,
(i) $f^* = q_1 ; f = f ; q_2$,
(ii) $(f ; q_2)^- = f ; q_2^-$,
(iii) $(q_1 ; f)^- = q_1^- ; f$.

Proof. (i). By 3.1(ii), $q_2 = f^\cup ; q_1 ; f$. Applying f on the left

$$f ; q_2 = f ; f^\cup ; q_1 ; f = e_1 ; q_1 ; f = q_1 ; f$$

by 3.1(i) and 1.2(i). Part (i) follows.

(ii) From 3.1(ii), $q_2^- \cdot (f^\cup ; q_1 ; f) = 0$ which, using (4), is equivalent to $(q_1 ; f) \cdot (f ; q_2^-) = 0$. Thus,

$$(f ; q_2)^- = (q_1 ; f)^- \geq f ; q_2^-.$$

On the other hand, $f ; f^\cup = e_1$ implies $1 = 1' ; 1 \leq f ; f^\cup ; 1 \leq f ; 1$ so

$$1 = f ; 1 = f ; (q_2^- + q_2) = f ; q_2^- + f ; q_2$$

which gives $(f ; q_2)^- \leq f ; q_2^-$.

(iii) The proof is similar to (ii). □

LEMMA 3.4. *For* $f \in \mathrm{Ism}(q_1, e_1 : q_2, e_2)$,
(i) $\pi_l(f^*) = q_1$,
(ii) $\pi_r(f^*) = q_2$,
(iii) $(f^*)^\sim = (f^\cup)^*$,
(iv) if $g \in \mathrm{Ism}(q_2, e_2 : q_3, e_3)$, then $(f ; g)^* = f^* ; g^*$.

Proof. (i) $(f^*)^- ; (f^*)^\cup = (q_1 ; f)^- ; (q_1 ; f)^\cup = q_1^- ; f ; f^\cup ; q_1^\cup = q_1^- ; e_1 ; q_1^\cup = q_1^- ; q_1^\cup = q_1^-$ using 3.3(i), 3.3(iii), 3.1(i), 1.2(i) and $\pi_l(q_1) = q_1$.

(ii) Similar to the proof of (i).

(iii) Since $f^\cup \in \mathrm{Ism}(q_2, e_2 : q_1, e_1)$, $(f^\cup)^* = q_2 ; f^\cup ; q_1$. Now,

$$(f^*)^{\sim -} = q_2^- ; (f ; q_2)^\cup = q_2^- ; q_2^\cup ; f^\cup = q_2^- ; f^\cup = (q_2 ; f^\cup)^- = (f^\cup)^{*-}$$

using 2.1(i), part (ii), 3.3(i), $\pi_l(q_2) = q_2$, 3.3(iii) and 3.3(i).

(iv). First observe that $f ; g \in \mathrm{Ism}(q_1, e_1 : q_3, e_3)$. Then, by 3.3(i), $(f ; g)^* = q_1 ; f ; g ; q_3 = f^* ; g^*$. □

The main result of this section characterizes invertible elements in a RA $\mathfrak{A}$ in terms of isomorphism elements.

THEOREM 3.5. *If q_i is a partial order element with respect to an equivalence element e_i $(i=1,2)$, the map that sends $f \to f^*$ is a bijection of* Ism $(q_1, e_1 : q_2, e_2)$ *onto $G(q_1, q_2)$. If $q_1 = q_2$, the map is a group isomorphism* Aut $(q_i, e_1) \cong G(q_1)$.

Proof. The second statement easily follows from the first using 3.2, 3.3, and 3.4 so we prove the first. Because

$$f^* ; f^{*\smile} = f^* ; (f^{\cup})^* = (f ; f^{\cup})^* = e_1^* = q_1$$

and similarly $f^{*\smile} ; f^* = q_2$, we have $f^* \in G(q_1, q_2)$. To prove the * map is one–one assume $f^* = g^*$. Then

$$f^{\cup} ; g \leq (f ; q_2)^{\cup} ; g = (g ; q_2)^{\cup} ; g = q_2^{\cup} ; g^{\cup} ; g = q_2^{\cup} ; e_2 = q_2^{\cup}$$

and similarly $f^{\cup} ; g \leq q_2$. By 1.2(ii)

(6) $f^{\cup} ; g \leq q_2 \cdot q_2^{\cup} = e_2$

which implies that $g = e_2 ; g = f ; f^{\cup} ; g \leq f ; e_2 = f$. By a similar argument $f \leq g$ and it follows that * is one–one. It remains to show that the * map is onto $G(q_1, q_2)$. For an $a \in G(q_1, q_2)$ define $f = a \cdot a^{\smile\cup}$. It is immediate that $f^{\cup} = a^{\smile} \cdot a^{\cup}$. Statements (7), (11), and (15) below show that f is the desired element.

(7) f is a bijection element from e_1 to e_2.

Clearly, $e_1 ; f \leq (e_1 ; a) \cdot (e_1 ; a^{\smile\cup}) = a \cdot a^{\smile\cup} = f$ because $a \in G(q_1, q_2)$. Hence, $e_1 ; f \leq f = 1' ; f \leq e_1 ; f$ so $f = e_1 ; f$. Similarly, $f ; e_2 = f$. Next,

(8) $f ; f^{\cup} = (a \cdot a^{\smile\cup}) ; (a^{\smile} ; a^{\cup}) \leq (a ; a^{\smile}) \cdot (a^{\smile\cup} ; a^{\cup}) = q_1 \cdot q_1^{\cup} = e_1$.

The inequality $\leq$ below is justified by Lemma 1.1.

(9) $e_1 = q_1 \cdot q_1^{\cup} = (a ; a^{\smile}) \cdot q_1^{\cup} \leq a ; ((a^{\cup} ; q_1^{\cup}) \cdot a^{\smile}) = a ; ((q_1 ; a)^{\cup} \cdot a^{\smile}$
$= a ; f^{\cup}$.

Using (9) and Lemma 1.1 we obtain

(10) $e_1 \leq (f ; a^{\cup}) \cdot q_1 \leq f ; ((f^{\cup} ; q_1) \cdot a^{\cup}) \leq f ; ((a^{\smile} ; q_1) \cdot a^{\cup}) = f ; f^{\cup}$.

From (8) and (10) we obtain $f ; f^{\cup} = e_1$. A similar argument gives $f^{\cup} ; f = e_2$ and this completes the proof of (7).

(11) $f \in$ Ism $(q_1, e_1 : q_2, e_2)$.

LEMMA 3.7. (i) *Each of the maps $c_{X,Y}$, c_X, c_Y, and $c_{Y,X}$ are one–one and onto with its inverse being given by the corresponding d. For example, $d_{X,Y}(c_{X,Y}(R)) = R$ and $c_{X,Y}(d_{X,Y}(a)) = a$ for all $R \subseteq X \times Y$ and $a \in E_1 ; \mathcal{Re}(Z) ; E_2$.*

(ii) *The c's (and also the d's) preserve composition. For example, for $S \subseteq X \times X$, $R \subseteq X \times Y$, and $T \subseteq Y \times X$, $c_X(S) ; c_{X,Y}(R) = c_{X,Y}(S \circ R)$ and $c_{X,Y}(R) ; c_{Y,X}(T) = c_X(R \circ T)$, etc.*

(iii) *$c(R^-) = c(R)^-$ and $c(R^{-1}) = c(R)^\cup$ for appropriate subscripts.*

From 3.7, 2.1, 1.4 and (16)–(19) we immediately obtain

LEMMA 3.8. *For $R \subseteq X \times Y$*

(i) $\pi_l(R) = d(\pi_l(c(R)))$,

(ii) $\pi_r(R) = d(\pi_r(c(R)))$,

(iii) $R^\sim = d((c(R))^\sim)$,

(iv) R is invertible iff $c(R)$ is invertible. Moreover, if R is invertible, its unique inverse is $R^\sim$.

It follows from 3.8 that if π_1 is a quasi-ordering on X and π_2 is a quasi-ordering on Y, then $R \in G(\pi_1, \pi_2)$ iff $c(R) \in G(c(\pi_1), c(\pi_2))$. Applying 3.8 to 1.5 and 1.6 we obtain properties of $\pi_l(R)$ and $\pi_r(R)$.

LEMMA 3.9. *For $R \subseteq X \times Y$,*

(i) *$\pi_l(R)$ is the largest solution S to $S \circ R = R$,*

(ii) *$\pi_r(R)$ is the largest solution S to $R \circ S = R$,*

(iii) *$\pi_l(R)$ (respectively, $\pi_r(R)$) is a quasi-ordering on X (respectively, Y).*

The units $\pi_l(R)$ and $\pi_r(R)$ where $R \subseteq X \times Y$ can also be characterized by the following formulas:

(20) $(x_1, x_2) \in \pi_l(R)$ iff $\forall y \in Y((x_2, y) \in R \Rightarrow (x_1, y) \in R)$

(21) $(y_1, y_2) \in \pi_r(R)$ iff $\forall x \in X((x, y_1) \in R \Rightarrow (x, y_2) \in R)$

In particular, (20) and (21) imply that if $F \subseteq X \times Y$ is a function,

(22) $\pi_l(F) = \ker(F) \cup X \times (X - \mathrm{Dom}(F))$ and

(23) $\pi_r(F) = I_Y \cup (Y\text{-}\mathrm{Ran}(F)) \times Y$.

If $F \subseteq X \times Y$ is a bijection of X onto Y, then (22) and (23) imply that $\pi_l(F) = I_X$ and $\pi_r(F) = I_Y$. Hence, in $\mathcal{R}e(Z)$, $\pi_l(c(F)) = E_1$ and $\pi_r(c(F)) = E_2$, i.e., $c(F) \in R(E_1, E_2)$. Also, F is invertible and, using (18), its quasi-inverse $F^{\sim} = F^{-1}$. Using 3.8(iv), it follows that $c(F)$ is a bijection element from E_1 to E_2. Conversely, if f is a bijection element from E_1 to E_2, $f \in R(E_1, E_2)$ so $f = c(F)$ for some $F \subseteq X \times Y$ by 3.7(i). The condition $f^{\cup} \circ f = E_2$ implies that F is a function whose range is Y while $f \circ f^{\cup} = E_1$ implies that F is one–one and its domain is X. Thus, F is a bijection from X onto Y. The argument above establishes the first statement in the lemma below. The second statement follows from the first and the fact that a bijection $F: X \to Y$ that satisfies the property $F^{-1} \circ \pi_1 \circ F = \pi_2$ is an isomorphism of $\langle X, \pi_1 \rangle$ onto $\langle Y, \pi_2 \rangle$. The set of isomorphisms of $\langle X, \pi_1 \rangle$ onto $\langle Y, \pi_2 \rangle$ is denoted by Ism (π_1, π_2).

LEMMA 3.10. *Suppose $F \subseteq X \times Y$, $f = c(F)$, π_1 (respectively, π_2) is a quasi-ordering on X (respectively, Y), and $q_i = c(\pi_i)$ for $i = 1, 2$. Then*
(i) *f is a bijection element from E_1 to E_2 iff F is a bijection from X onto Y,*
(ii) *$f \in$ Ism $(q_1, E_1 : q_2, E_2)$ iff $F \in$ Ism (π_1, π_2).*

Now suppose π_1 is a quasi-ordering on X, π_2 is a quasi-ordering on Y, and $q_i = c(\pi_i)$ for $i = 1, 2$. With respect to $e_i = q_i \cap q_i^{\cup}$, q_i is a partial order element and, by 3.6(i), $G(q_1, q_2) \cong$ Ism $(q_1, e_1 : q_2, e_2)$. Using the definition of e_1, $e_1 \circ \mathcal{R}e(X \times Y) \circ e_1 \cong \mathcal{R}e(X')$ where $X' = X/(\pi_1 \cap \pi_1^{\cup})$ and q_1 corresponds to a partial ordering $^*\pi_1$ on X'. Similarly, q_2 corresponds to a partial ordering $^*\pi_2$ on $Y' = Y/(\pi_2 \cap \pi_2^{\cup})$.

Using the correspondence between Ism $(^*\pi_1, {}^*\pi_2)$ and Ism $(q_1, e_1 : q_2, e_2)$ given in 3.10, we obtain from 3.5

THEOREM 3.11. *For quasi-orderings π_i on X_i (for $i = 1, 2$), there is a natural bijection between $G(\pi_1, \pi_2)$ and* Ism $(\langle {}^*X_1, {}^*\pi_1 \rangle, \langle {}^*X_2, {}^*\pi_2 \rangle)$ *where $^*\pi_i$ is the partial ordering induced by π_i on $^*X_i = X_i/(\pi_i \cap \pi_i^{-1})$. In particular, if $X_1 = X_2 = X$ and $\pi_1 = \pi_2 = \pi$, $G(\pi) \cong$* Aut $(\langle {}^*X, {}^*\pi \rangle)$.

These results can be obtained directly without using relation algebras. To do so, first show that $G(\pi_1, \pi_2) \cong G(^*\pi_1, {}^*\pi_2)$ where $^*\pi_i$ is the partial ordering induced by π_i $(i = 1, 2)$. Then show that Ism $(^*\pi_1, {}^*\pi_2) \cong G(^*\pi_1, {}^*\pi_2)$ using the map that sends an isomorphism $F: \langle {}^*X_1, {}^*\pi_1 \rangle \to \langle {}^*X_2, {}^*\pi_2 \rangle$ to the relation $^*\pi_1 \circ F \circ {}^*\pi_2$.

4. A topological characterization of $Q(\mathcal{R}e(X))$.

As mentioned at the start of section 2 the collection $Q(\mathfrak{A})$ of all quasi-order elements in a complete RA $\mathfrak{A}$ forms a complete lattice. For short, we let $Q(X) = Q(\mathcal{R}e(X))$ the lattice of all quasi-orderings on a set X. In this section the lattice $Q(X)$ is described. As a corollary we characterize the units $\pi_l(R)$ and $\pi_r(R)$ of a relation $R \subseteq X \times Y$.

For a relation $R \subseteq X \times Y$ the operation $R^{\dagger} = R^{--1} \subseteq Y \times X$ is a concrete version of the † operation in a general RA. Below a basic quasi-ordering is associated with each subset of X.

DEFINITION 4.1. For $A \subseteq X$, let $\pi(A) = (A \times (X - A))^{\dagger}$.

Equivalently, we could define $\pi(A) = (A \times X) \cup (X \times (X - A))$. It is easy to verify that $\pi(A)$ is a quasi-ordering on X.

THEOREM 4.2. *Every quasi-ordering π on X is an intersection of $\pi(A)$'s.*

Proof. Because $X \times X = \pi(X)$ we may assume $\pi \neq X \times X$. For $(x, y) \notin \pi$ let $A_{x,y} = \{z \in X : (z, y) \in \pi \text{ and } (x, z) \notin \pi\}$. Then, for $(x, y) \notin \pi$, it easily follows that

(24) $\pi \subseteq \pi(A_{x,y})$, and

(25) $(x, y) \notin \pi(A_{x,y})$

from which we obtain $\pi = \bigcap \{\pi(A_{x,y}) : (x, y) \notin \pi\}$. □

COROLLARY 4.3. *The $\pi(A)$'s for $A \neq 0, X$ are exactly the maximal quasi-orderings on X.*

Proof. Every maximal quasi-ordering on X has the form $\pi(A)$ by 4.2. On the other hand, if $\pi(A) \subseteq \pi(B)$ and $B \neq 0, X$, then (applying †) we see that $A \supseteq B$ and $X - A \supseteq X - B$, so $A = B$. Thus, $\pi(A)$ is maximal whenever $A \neq 0, X$. □

A topology on X is called a *$\cap$-topology* if it is closed under arbitrary intersections. The collection of all $\cap$-topologies on X form a complete lattice denoted by $\mathcal{T}_{\cap}(X)$. A $\cap$-topology $\mathcal{T}$ is said to be *generated by* a collection $\mathcal{A}$ of subsets of X if $\mathcal{T}$ is the smallest $\cap$-topology that contains $\mathcal{A}$. It is easily seen that

LEMMA 4.4. *A $\cap$-topology $\mathcal{T}$ is generated by a collection $\mathcal{A}$ iff every element of $\mathcal{T}$ is a union of intersections of members of $\mathcal{A}$.*

A topology associated with $\pi \in Q(X)$ is defined by $\mathcal{T}_\pi = \{A \subseteq X : \pi \subseteq \pi(A)\}$. The next lemma shows that every quasi-order determines a $\cap$-topology.

LEMMA 4.5. *For $\pi \in Q(X)$ $\mathcal{T}_\pi$ is a $\cap$-topology on X.*

Proof. Clearly $0, X \in \mathcal{T}_\pi$. The inclusion

$$(X - \bigcup_i A_i) \times (\bigcup_i A_i) \subseteq \bigcup_i (X - A_i) \times A_i$$

implies

$$(26)\quad \bigcap_i \pi(A_i) \subseteq \pi(\bigcup_i A_i)$$

which shows that $\mathcal{T}_\pi$ is closed under arbitrary unions. Similary,

$$\begin{aligned}(X - \bigcap_i A_i) \times (\bigcap_i A_i) &= \bigcup_i (X - A_i) \times \bigcap_i A_i \\ &= \bigcup_i [(X - A_i) \times \bigcap_i A_i] \\ &\subseteq \bigcup_i [(X - A_i) \times A_i]\end{aligned}$$

yields

$$(27)\quad \bigcap_i \pi(A_i) \subseteq \pi(\bigcap_i A_i)$$

which implies that $\mathcal{T}_\pi$ is closed under arbitrary intersections. □

THEOREM 4.6. *The correspondence $\pi \to \mathcal{T}_\pi$ is an anti-isomorphism of $Q(X)$ onto $\mathcal{T}_\cap(X)$.*

Proof. The map is one–one by 4.2 and clearly $\pi \subseteq \pi'$ implies $\mathcal{T}_{\pi'} \subseteq \mathcal{T}_\pi$, so it suffices to show the map is onto. Suppose $\mathcal{T}$ is a $\cap$-topology on X and let $\pi_{\mathcal{T}} = \cap \{\pi(B) : B \in \mathcal{T}\}$. Clearly $\pi_{\mathcal{T}}$ is a quasi-ordering and $\mathcal{T} \subseteq \mathcal{T}_{\pi_{\mathcal{T}}}$ because $A \in \mathcal{T}$ implies $\pi_{\mathcal{T}} \subseteq \pi(A)$. For each $B \in \mathcal{T}$

$$(28)\quad \pi(B) = \bigsqcup_{y \notin B} (X \times \{y\}) \sqcup \bigsqcup_{y \in B} (B \times \{y\})$$

so it follows that

$$(29)\quad \pi_{\mathcal{T}} = \bigsqcup_{y \in X} \left(\bigcap \{B : y \in B \in \mathcal{T}\}\right) \times \{y\}.$$

Now, suppose $A \subseteq X$, $\pi(A) \supseteq \pi_{\mathcal{T}}$, and $x \in A$. Comparing (28) for A and (29) it follows that

$$x \in \bigcap \{B : x \in B \in \mathcal{T}\} \subseteq A.$$

Thus, $A \in \mathcal{T}$ and it follows that $\mathcal{T}_{\pi_{\mathcal{T}}} = \mathcal{T}$ as desired. □

The proof that the map in 4.6 is onto has several consequences.

COROLLARY 4.7. *Suppose π is a quasi-ordering on X, $\mathcal{T}_\pi$ the associated $\cap$-topology, and $\pi = \sqcup_{y \in X} A_y \times \{y\}$. Then*
(i) *$\mathcal{T}_\pi$ is generated by $\{A_{x,y} : (x, y) \notin \pi\}$.*
(ii) *$\{A_y : y \in X\}$ is a basis for $\mathcal{T}_\pi$.*
(iii) *A_y is the smallest open set of $\mathcal{T}_\pi$ that contains y.*

Proof. (ii) For $B \in \mathcal{T}_\pi$. $\sqcup_y A_y \times \{y\} \subseteq (B \times X) \sqcup (X \times (X - B))$, so if $y \in B$, $y \in A_y \subseteq B$. □

The following generalizes the correspondence between finite posets and finite T_0-spaces given in [1]. The treatment in [1] uses A_y as the closure of y (cf., 4.7(iii) above).

THEOREM 4.8. *π is a partial ordering of X iff $\mathcal{T}_\pi$ is a T_0-topology.*

Proof. Suppose π is a partial ordering on X. For $x \neq y$, either $(x, y) \notin \pi$ or $(y, x) \notin \pi$. Thus, either $A_{x,y} \in \mathcal{T}_\pi$ or $A_{y,x} \in \mathcal{T}_\pi$ which shows $\mathcal{T}_\pi$ is a T_0-topology because $y \in A_{x,y}$ and $x \notin A_{x,y}$. Conversely, suppose $\mathcal{T}_\pi$ is a T_0-topology and consider $x \neq y$. By 4.7(iii) either $x \notin A_y$ or $y \notin A_x$. If $x \notin A_y$, then $(y, x) \in (A_y \times (X - A_y))^\dagger = \pi(A_y)$. Similarly, $y \notin A_x$ implies $(x, y) \in \pi(A_x)$. Thus, π is a partial ordering. □

For a given relation R the final result characterizes $\pi_l(R)$ and $\pi_r(R)$ using topologies.

THEOREM 4.9. *Suppose π_1 and π_2 are quasi-orderings on X and Y respectively and $R \subseteq X \times Y$. Further, suppose*

$$R = \bigsqcup_{y \in Y} A_y \times \{y\} = \bigsqcup_{x \in X} \{x\} \times B_x.$$

Then (i) $\pi_1 = \pi_l(R)$ *iff* $\{A_y : y \in Y\}$ *generates* $\mathcal{T}_{\pi_1}$.
(ii) $\pi_2 = \pi_r(R)$ *iff* $\{Y - B_x : x \in X\}$ *generates* $\mathcal{T}_{\pi_2}$.

Proof. (i) Since $R^\dagger = \sqcup_y \{y\} \times (X - A_y)$,

$$\pi_l(R)^\dagger = R \circ R^\dagger = \bigcup_y A_y \times (X - A_y) = \bigcup_y \pi(A_y)^\dagger.$$

Applying † gives $\pi_l(R) = \cap_y \pi(A_y)$, so $\{A_y : y \in Y\}$ generates $\mathscr{T}_{\pi_l}(R)$ by the argument in 4.6. Thus, (i) follows. The proof of (ii) is similar. □

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