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WEAKLY REGULAR TREES AND THEIR COLOR ALGEBRAS

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ABSTRACT: A finite semilattice X is weakly regular if it admits a length function ℓ such that for all $r,s,t \leq m$

$$\exists x,y,z\in X_m\ [xy\in X_r \wedge xz \in X_s \wedge yz\in X_t] \text{ implies } \forall x,y\in X_m\ [xy\in X_r \rightarrow \exists z\in X_m\ [xz\in X_s \wedge yz\in X_t]]$$

where $X_i = \{x\in X:\ell(x) = i\}$ for all i and m is the maximum value of ℓ. Every regular semilattice is weakly regular. A weakly regular semilattice determines a color scheme using a construction of Delsarte. Two weakly regular semilattices are equivalent if the polygroups (hypergroups) associated with their color schemes are isomorphic. A geometrical characterization of weakly regular trees is given. It is shown that every weakly regular tree is equivalent to a regular tree and the polygroups of these systems are completely determined.

In [3] Delsarte introduced the concept of a regular semilattice and showed how to construct an association scheme from such a semilattice. Association schemes are color schemes in the sense of [2]. We introduce the notion of a weakly regular semilattice and show that Delsarte's construction produces a color scheme $\mathcal{C}(X)$ from a finite semilattice X with a length function if and only if the semilattice is weakly regular. Unfortunately, it is not easy to tell whether a semilattice is weakly regular. It is desirable to characterize weak regularity by conditions of a geometrical nature similar to the conditions used by Delsarte for regularity. In section 2 such conditions are given when the semilattice is a tree. Two weakly regular semilattices X and Y are color equivalent if the color algebras associated with $\mathcal{C}(X)$ and $\mathcal{C}(Y)$ are isomorphic. We show that every weakly regular tree is color equivalent to a regular tree and completely describe the color algebras of regular trees.

For unexplained notation and terminology the reader should consult [2] and [3].

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1. Weakly Regular Semilattices

A finite poset $<X,\leq>$ is a (meet) semilattice if every two points $x,y\epsilon X$ has a greatest lower bound denoted by xy. Let 0 denote the least element of X. A *length function* on a finite semilattice X is a function $\ell:X\rightarrow\omega$ such that $\ell(0) = 0$ and for every $x,y\epsilon X$ with $x < y$ there exist $u\epsilon X$ $x < u \leq y$ and $\ell(u) = \ell(x)+1$. A semilattice admits at most one length function.

The construction below of a relational system $\mathcal{V}(X)$ from a finite semilattice that admits a length function was introduced in [3] as a way to construct association schemes.

Let m denote the maximum value of $\ell(x)$ and define fibers $X_0,\ldots X_m$ by

$$X_i = \{x\epsilon X : \ell(x) = i\}.$$

Let $e_0 = m > e_1 > \ldots > e_n$ be a list of the distinct values of $\ell(xy)$ for $x,y\epsilon X_m$. For $i \leq n$ define the relation $R_i \subseteq X^2_m$ by

$$R_i = \{(x,y)\ \epsilon X^2_m : \ell(xy) = e_i\}$$

and set $\mathcal{V}(X) = <X_m,R_0,\ldots,R_n>$.

A binary relational system $(V,R_0,\ldots R_n)$ is a *(symmetric) n-color scheme* (cf., [2]) if

(i) $\{R_0,\ldots,R_n\}$ is a partition of V, $R_i\neq\emptyset$ for all i, and $R_0 = I_V$

(ii) for all $r,s,t \leq n$, $R_r\cap(R_s|R_t)\neq\emptyset$ implies $R_r\subseteq R_s|R_t$.

Association schemes are color schemes but not every color scheme is an association scheme. Color schemes arise in the study of representations of relations algebras. We want to characterize semilattices X for which $\mathcal{V}(X)$ is a color scheme.

Definition 1. A finite semilattice $<X,\leq>$ that admits a length function is *weakly regular* if for all $r,s,t \leq m$

(1.1) $\exists x,y,z\epsilon X_m[xy\epsilon X_r \wedge xz\epsilon X_s \wedge yz\epsilon X_t]$ implies $\forall x,y\epsilon X_m$ $[xy\epsilon X_r \rightarrow \exists z\epsilon X_m\ [xz\epsilon X_s \wedge yz\epsilon X_t]]$

Because (1.1) is a restatement of (ii) in the definition of color scheme the following characterization is obvious.

Theorem 2. *$\mathcal{V}(X)$ is a color scheme iff $<X,\leq>$ is weakly regular.*

To see that Theorem 2 is a weak form of Theorem 7 of [3] we need to see that a regular semilattice is weakly regular.

Definition 3 [3]. A semilattice $<X, \leq>$ that admits a length function is *regular* if
(1) for $y \epsilon X_m$, $z \epsilon X_r$ with $z \leq y$ $|\{u \epsilon X_s : z \leq u \leq y\}|$ if a constant $\mu(r,s)$.
(2) for $u \epsilon X_s$, $|\{z \epsilon X_r : z \leq u\}|$ is a constant $\nu(r,s)$.
(3) for $a \epsilon X_r$, $y \epsilon X_m$ with $ay \epsilon X_j$ $|\{(b,z) \epsilon X_s \times X_m : b \leq zy , a \leq z\}|$ is a constant $\pi(j,r,s)$.

Proposition 4. *A regular semilattice is weakly regular.*

Proof. Suppose $<X, \leq>$ is a regular semilattice. Let $D_K \epsilon \mathbb{R}(X_m, X_m)$ denote the adjacency matrix of the graph $<X_m, R_K>$, i.e.,

$$D_k(x,y) = \begin{cases} 1 & \text{if } \ell(xy) = e_k \\ 0 & \text{otherwise} \end{cases}.$$

The proof of Theorem 7 in [3] that $<X_m, R_O, \ldots, R_n>$ is an association scheme amounts to showing that the vector space generated by $D_O, \ldots, D_n$ is a multiplicative algebra. In particular, it is shown that

$$D_s D_t = \sum_k p_{st}^k D_k$$

for real numbers p_{st}^k. The hypothesis of condition (1.1) asserts that $(D_s D_t)(x,y) = 1$ and $D_r(x,y) = 1$ for some $x,y \epsilon X_m$. Since $D_r(x,y) = 1$ implies $D_k(x,y) = 0$ for all $k \neq r$, the algebra equation shows that $p_{st}^r > 0$. Now suppose $x,y \epsilon X_m$ with $D_r(x,y) = 1$. Since $p_{st}^r > 0$ and $D_k(x,y) = 0$ for all $k \neq r$, $(D_s D_t)(x,y) > 0$. Thus, there exist $z \epsilon X_m$ with $xz \epsilon X_s$ and $zy \epsilon X_t$ as desired. □

2. Weakly Regular Trees

A *tree* is a semilattice $<X, \leq>$ such that $(x] = \{y : y \leq x\}$ is well ordered by $\leq$ for all $x \epsilon X$. The goal of this section is to characterize weakly regular trees by conditions similar to 3(1), 3(2) and 3(3).

We assume throughout that $<X, \leq>$ is a tree that admits a length function ℓ. For $i \leq m$ let

$$\overline{X}_i = \{x \epsilon X_i : x \leq y \text{ for some } y \epsilon X_m\}.$$

Clearly $\overline{X}_m = X_m$ and if $x,y \epsilon X_m$ then $xy \epsilon \overline{X}_r$ for $r = \ell(xy)$. Also, if $b \epsilon (x] \cap X_r$ where $x \epsilon X_m$ then $b \epsilon \overline{X}_r$. For $x \epsilon X_s$ the *outdegree* of x is the number $\delta(x) = |\{y \epsilon \overline{X}_{s+1} : y \geq x\}|$.

Consider the two conditions:

T1. For $r \leq m$, if $\delta(x) = 2$ for some $x \epsilon \overline{X}_r$ then $\delta(x) = 2$ for every $x \epsilon \overline{X}_r$.

T2. For $r \leq m$, if $\delta(x) \geq 3$ for some $x \epsilon \overline{X}_r$ then $\delta(x) \geq 3$ for every $x \epsilon \overline{X}_r$.

We consider two alternatives to T1. Namely,

T1$'$. For $r \leq m$, $\delta(x) \geq 2$ for some $x \epsilon \overline{X}_r$ implies $\delta(x) \geq 2$ for all $x \epsilon \overline{X}_r$.

T1$''$. For $r \leq m$, $\delta(x) = 1$ for some $x \epsilon \overline{X}_r$ implies $\delta(x) = 1$ for all $x \epsilon \overline{X}_r$.

Lemma 5. *Suppose X is a tree that admits a length function.*
(1) Condition T1$'$ holds iff condition T1$''$ holds.
(2) If X satisfies T2, then T1$'$ holds iff T1 holds.

Proof. (1) ($\Rightarrow$) Suppose $\delta x = 1$ and $\delta y \neq 1$ for some $x, y \epsilon \overline{X}_r$.
Since $\delta y \geq 1$, it follows that $\delta y \geq 2$ which, by T1$'$, implies that $\delta x \geq 2$ contradicting $\delta x = 1$.

($\Leftarrow$) Suppose $\delta x \geq 2$ for some $x \epsilon \overline{X}_r$ and $2 \nleq \delta y$ for some $y \epsilon \overline{X}_r$. Thus, $\delta y = 1$ which implies, by T1$''$, that $\delta x = 1$ contradicting $\delta(x) \geq 2$.

(2) T1$'$ clearly follows from T1 and T2. For the converse, suppose $\delta x = 2$ for some $x \epsilon \overline{X}_r$. T1$'$ implies $\delta y \geq 2$ for all $y \epsilon \overline{X}_r$. If $\delta y > 2$ for some $y \epsilon \overline{X}_r$, then T2 implies $\delta x > 2$; thus, $\delta y = 2$ for all $y \epsilon \overline{X}_r$. □

Conditions T1 and T2 mean that δ is "constant on fibers" for small values.

Theorem 6. *A tree $<X, \leq>$ is weakly regular iff T1 and T2 hold.*

Proof. ($\Rightarrow$) Suppose $<X, \leq>$ is weakly regular. To verify T1$'$, suppose $x, y \epsilon \overline{X}_s$ with $\delta(x) \geq 2$; say $x', x'' \epsilon X_{s+1}$ $x'x'' = x$. Choose $x_1, x_2 \epsilon X_m$ extending x', x'' and observe that $x_1x_2 = x \epsilon \overline{X}_s$. Thus, the hypothesis of (1.1) holds where $s = t$ and $r = 0$. Choose $y_1 \epsilon X_m$ with $y_1 \geq y$. Then by (1.1) there exist $z \epsilon X_m$ such that $y_1 z \epsilon \overline{X}_s$. Since $(y_1]$ is well ordered $(y_1] \cap X_s = \{y\}$ and so $y_1 z = y$. The elements y' and y'' defined by $\{y'\} = (y_1] \cap X_{s+1}$ and $\{y''\} = (z] \cap X_{s+1}$ are distinct covers of y so $\delta(y) \geq 2$. This shows that T1$'$ holds.

To verify T2, suppose, $x, y \epsilon \overline{X}_r$ and $\delta(x) \geq 3$. Hence there exist distinct elements $x_1, x_2, x_3 \epsilon X_m$ such that $x_1x_2 = x_1x_3 = x_2x_3 = x$. By T1$'$ $\delta y \geq 2$ so $y = y_1y_2$ for some $y_1, y_2 \epsilon X_m$. Applying (1.1) with $r = s = t$ there exist $z \epsilon X_m$ with $y_1 z \epsilon X_r$ and $y_2 z \epsilon X_r$. Since $\{y\} = (y_1] \cap X_r = (y_2] \cap X_r$, $y_1 z = y_2 z = y$ and therefore $\delta y \geq 3$. By Lemma 5, T1 and T2 hold.

($\Leftarrow$) Assume T1,T2 hold and there exist $x_1, y_1, z_1 \epsilon X_m$ with $x_1y_1 \epsilon \overline{X}_r$, $x_1z_1 \epsilon \overline{X}_s$ and $y_1z_1 \epsilon \overline{X}_t$. Suppose $x, y \epsilon X_m$ with $xy \epsilon \overline{X}_r$. We consider cases.

$r = s = t = m$.

Then $x = y$ and (1.1) obviously holds with $z = x$.

$r = m$ and $s = t$ $\neq$ m.

Then $x_1 = y_1$ and $\delta(x_1z_1) \geq 2$. Choose $u \in \overline{X}_s$ with $u \leq x = y$. T1′ implies $\delta u \geq 2$. Choose $z' \in X_{s+1}$ with $xz' = u$ and extend z' to $z \in X_m$. Then $xz = yz = u \in \overline{X}_s$ as desired.

$r \neq m$ and $s > r$.

Because $(z_1]$ is well ordered the unique element in $(z_1] \cap X_s$ is x_1z_1. Similarly $(x_1] \cap X_r = (z_1] \cap X_r = \{x_1y_1\}$. Therefore, $z_1y_1 = x_1y_1$ so $t = r$ in this case. Now suppose $x,y \in X_m$ with $xy \in \overline{X}_r$. Choose $b \in \overline{X}_s$, $b \leq x$. Since $\ell(x_1z_1) = s$, condition T1 implies there exist $z \in X_m$ with $xz = b \in \overline{X}_s$. Now, $yz = yx \in \overline{X}_r = \overline{X}_t$ as desired.

$r \neq m$ and $r > s$.

First, $t \geq s$ follows because $(x_1]$ is well ordered, $r > s$, $x_1y_1 \in X_r$ and $x_1z_1 \in X_s$. If $\ell(y_1z_1) = t > s$ then $\overline{X}_{\min\{r,t\}}$ contains an element $\leq x_1z_1$ which contradicts $\ell(x_1z_1) = s$; thus $t = s$. Now suppose $x,y \in X_m$ with $xy \in \overline{X}_r$ and choose $b \in \overline{X}_s$ with $b \leq x$. Since $\delta(x_1z_1) \geq 2$, $\delta b \geq 2$ so choose $z \in X_m$ with $xz = b \in \overline{X}_s$. Also, $yz = b \in \overline{X}_s = \overline{X}$ because $x'z' = b$ where z' is the unique element in $(z] \cap X_{s+1}$ and x' is the unique element in $(x] \cap X_{s+1} = (y] \cap X_{s+1}$.

$r = s \neq m$ and $t \neq m$ (since $t = m$ is obvious).

In this case x_1,y_1,z_1 are all distinct and $\{x_1y_1\} = (x_1] \cap \overline{X}_r = \{x_1z_1\} = (y_1] \cap \overline{X}_r = \{y_1z_1\}$. Therefore $t = r$ and $\delta(x_1y_1) \geq 3$. Hence $\delta(xy) \geq 3$. Choose $z' \in X_{r+1}$ distinct from the elements in $(x] \cap X_{r+1} \cup (y] \cap X_{r+1}$. Then $z'x = z'y = xy$. Extend z' to $z \in X_m$ to produce the desired conclusion. □

3. Classification of Weakly Regular Trees

A weakly regular semilattice X can be classified by a multi-valued algebra that indicates how the relations of $\mathscr{V}(X)$ compose. The notion of a color algebra and its properties can be found in [2].

Definition 7. (1) *The color algebra* of a symmetric n-color scheme $\mathscr{V} = \langle V, R_0, \ldots, R_n \rangle$ is the system $\mathscr{M}_{\mathscr{V}} = \langle \{0,\ldots,n\}, *, 0 \rangle$ where, for $r,s,t \in \{0,\ldots,n\}$, $*$ is defined by

$$s*t = \{r : R_r \subseteq R_s | R_t\}.$$

(2) Two weakly regular semilattices X and Y are *(color) equivalent* if $\mathscr{M}_{\mathscr{V}}(X) \cong \mathscr{M}_{\mathscr{V}}(Y)$.

Basically, the color equivalence of X and Y means that the color schemes they determine generate isomorphic algebras of relations (cf., [2]). By Theorems 8 and 9 the color algebras of weakly regular trees are completely described.

Theorem 8. *Every weakly regular tree is (color) equivalent to a regular tree.*

Proof. Given a weakly regular tree $<X, \leq>$ we construct another tree $<\hat{X}, \leq>$ by the following procedure.

Step 1: Delete all nodes in $X_r - \overline{X}_r$ for all r to form X'.

In the resulting tree every element in X'_r, $r \neq m$, extends to an element of X'_m, i.e., $\overline{X'_r} = X'_r$.

Step 2: Form the quotient tree $X'' = X'/\approx$ where $x \approx y$ iff (y covers x and $\delta x = 1$) or (x covers y and $\delta y = 1$).

T1$''$ holds in X' by Lemma 5, so X'' is obtained by identifying X'_r with X'_{r+1} when some element of X'_r has a unique cover. X'' satisfies T1 and T2 because $\delta x \geq 2$ and $x \approx y$ implies $x = y$. Because $\delta(x) \geq 2$ for every $x \epsilon X''$, not in X''_m, every element in X'' has the form yz for some $y,z \epsilon X''_m$.

Step 3: For each x, say in X''_r with $\delta x = n > 3$ choose distinct x_1,x_2,x_3 in X''_{r+1} which cover x and replace $[x) = \{y \epsilon X'' : y \geq x\}$ with

$$\{x\} \cup [x_1) \cup [x_2) \cup [x_3).$$

Let $\hat{X}$ be the resulting tree. Clearly $\hat{X}$ is weakly regular, δx depends only on the fiber that contains x (by T1 and T2) and for every x $\delta x = 2$ or $\delta x = 3$.

Claim 1. $\hat{X}$ and X are color equivalent.

It suffices to check that each step of the construction of $\hat{X}$ does not effect the color algebra associated with X. Clearly Step 1 and Step 2 does not change $\mathscr{M}\mathscr{V}(X)$ because we only remove elements that never appear as a product of elements from X_m. To see that Step 3 does not change the color algebra of $\mathscr{V}(X'')$ note that to compute a product in $\mathscr{M}\mathscr{V}(X'')$ we never need more that 3 elements $x_1,x_2,x_3 \in X''_m$ with x_1x_2, x_1x_3, $x_2x_3 \in X''_r$ and this case occurs only when $R_i \cap (R_i \mid R_i) \neq \emptyset$ with $e_i = r$.

Claim 2. $<\hat{X}, \leq>$ is a regular tree.

We need to check (1), (2), and (3) of definition 3.

(1) Given $y \epsilon \hat{X}_m$, $z \epsilon \hat{X}_r$ with $z \leq y$ there is exactly one point $u \epsilon \hat{X}_s$ such that $z \leq u \leq y$ because (y] is well ordered, ie., $\mu(r,s) = 1$.

(2) For similar reasons $\nu(r,s) = 1$.

(3) Given $a \epsilon \hat{X}_r, y \epsilon \hat{X}_m$ with $ay \epsilon X_j$; the value of $\pi(j,r,s)$ depends on the value of δ on the fibers above a. Choose $x_i \epsilon \hat{X}_{m-i}$ then observe that $|\{z \epsilon \hat{X}_m : z \geq a\}| = \delta(x_1) \bullet \ldots \bullet \delta(x_{m-r})$. For each $z \epsilon \hat{X}_m$ with $z \geq a$, $yz = ay$. There is only one element $b \epsilon \hat{X}_s$ such that $b \leq zy$ if $s \leq j$ and no elements if $s \not> j$. Hence

$$\pi(j,r,s) = \delta(x_1) \bullet \ldots \bullet \delta(x_{m-r}) \bullet d_{\leq}(s,j)$$

where $d_{\leq}(s,j) = \begin{cases} 1 & \text{if } s \leq j \\ 0 & \text{otherwise} \end{cases}$. □

As observed in the proof of Theorem 8 the regular trees $\hat{X}$ are determined by the values of δ on the fibers. We call a tree $\hat{X}$ *reduced* and say that a reduced regular tree has *type* $<\tau_1,\ldots,\tau_n>$ if $\tau_i = \delta(x_i) - 2$ where x_i is some element of X_{m-i}. T1 and T2 imply that the type is independent of the choice of x_i' s.

The color algebra of a reduced regular tree will be described below. The description uses the notion from [1] of the extension $\mathcal{M}[\mathcal{N}]$ of a polygroup $\mathcal{M}$ by a polygroup $\mathcal{N}$. The polygroup T is the system whose product table is

	0	1
0	0	1
1	1	{0,1}

Theorem 9. *Suppose* $<X,\leq>$ *is a reduced regular tree with type* $<\tau_1,\ldots,\tau_m>$. *Then* $\mathcal{M}_{\mathcal{V}}(X) \cong \mathcal{M}_1[\mathcal{M}_2[\ldots[\mathcal{M}_m]\ldots]]$ *where for all i* $\mathcal{M}_i = \mathbb{Z}_2$ *if* $\tau_i = 0$ *and* $\mathcal{M}_i =$ T if $\tau_i = 1$.

Proof. Suppose the distinct values of $\ell(xy)$ for $x,y \epsilon X_m$ are $m = e_0 > e_1 > \ldots > e_m = 0$, i.e., $e_i = r$ iff $i + r = m$. Let the universe of $\mathcal{M}_{\mathcal{V}}(X)$ be $\{0,\ldots,m\}$ where i corresponds to R_i. The product $*$ in $\mathcal{M}_{\mathcal{V}}(X)$ is described by the following conditions:

(1) $i*j = j*i$ for all i,j

(2) $0*i = i$ for all i (because R_0 is the identity).

(3) $i*i = \begin{cases} \{0,\ldots,i-1\} & \text{if } \tau_i = 0 \\ \{0,\ldots,i\} & \text{if } \tau_i = 1 \end{cases}$ for $i = 1,\ldots,m$

Suppose $e_i = r$ and $(x,y)\epsilon R_j$ where $j < i$. Let $b \epsilon X_r$, $b \leq x$. $\delta(b) \geq 2$ so choose $z \epsilon X_m$ with $xz = b$. (Note that also $yz = b$ since $xy \epsilon X_{m-j}$ and $j < i$.) Thus, (x,z), (z,y) ϵR_i and so $j \epsilon i*i$ in both cases. Now suppose $(x,y)\ \epsilon R_i$ and $b = xy \epsilon X_r$. Then there exist $z \epsilon X_m$ with (x,z), $(z,y)\ \epsilon\ R_i$ iff $\delta(b) \geq 3$.

Hence $i \epsilon i*i$ if $\delta(x_i) = 3$ but not otherwise.

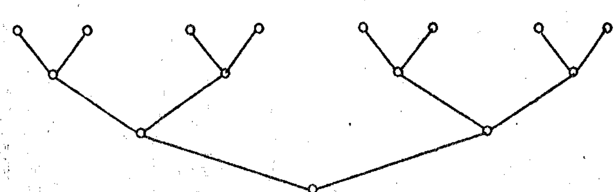
(4) for $i,j \epsilon \{1,\ldots,m\}$ and $i \neq j$, $i*j = \max\{i,j\}$.

We suppose i < j and show $i*j = j$. Suppose $(x,y)\epsilon R_j$ where $e_j = r$ and choose $b \epsilon (x] \cap X_{m-i}$. Since $\delta(b) \geq 2$ there exist $z \epsilon X_m$ $xz = b$, i.e., $(x,z)\epsilon R_j$. But $yz = xy$ since $i < j$; so $(z,y)\epsilon R_j$ and therefore $R_j \cap (R_i|R_j) \neq \emptyset$. Thus, $j \epsilon i*j$. Now if $(x,y) \epsilon R_k \cap (R_i|R_j)$ there exist z with $(x,z)\epsilon R_i$ and $(z,y)\epsilon R_j$. Then $xz > zy$ because $i < j$, $xz \epsilon X_{m-i}$ and $zy \epsilon X_{m-j}$. It follows that $xy = zy \epsilon X_{m-j}$, i.e., $(x,y)\epsilon R_j$. This shows that if $k \epsilon i*j$, then $k = j$. Thus, $i*j = j$ as desired.

The stated description of $\mathcal{M}\mathcal{Q}(X)$ follows from properties (1)-(4) above and the product definition in the indicated extension. □

Figures 1, 2, and 3 illustrate the color algebras associated with a few simple regular trees.

It would be useful to characterize weakly regular semilattices by conditions like T1 and T2. Also, is every weakly regular semilattice color equivalent to a regular semilattice?

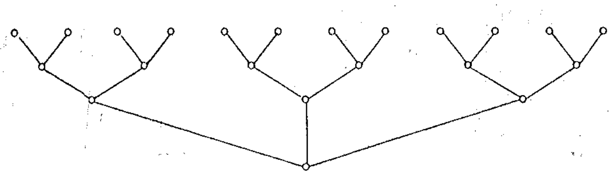


	0	1	2	3
0	0	1	2	3
1	1	0	2	3
2	2	2	01	3
3	3	3	3	012

Color algebra

Full binary tree of type <0,0,0>

Fig. 1

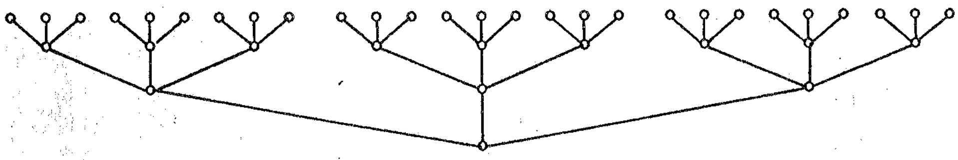


	0	1	2	3
0	0	1	2	3
1	1	0	2	3
2	2	2	01	3
3	3	3	3	0123

Color algebra

Tree of type <0,0,1>

Fig. 2



	0	1	2	3
0	0	1	2	3
1	1	01	2	3
2	2	2	012	3
3	3	3	3	0123

Tree of type < 1,1,1 > and its color algebra

Fig. 3

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