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SOME PROBLEMS ON HYPERGROUPS[1]

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This paper describes a few open problems concerning special classes of hypergroups. Solutions to these problems would not only provide valuable insight into the structure of hypergroups, but also strengthen the connections between hypergroups and other mathematical structures. This survey is not intended to be comprehensive, but, instead, to reflect the main areas of interest of the author. Other aspects of the theory of hypergroups will be adequately treated in contributions by other authors. We will mainly deal with hypergroups that are very close to ordinary groups, namely, with polygroups ([6]). We recall that a polygroup is the same as a quasi–canonical hypergroup in the sense of Bonansinga and Corsini [5].

1. SPECIAL CLASSES

Hypergroups (and polygroups) arose from attempts to isolate and clarify ideas from topology and group theory. More recently connections have developed with combinatorics and algebraic logic. Some of the algebras and classes of algebras that naturally arise are described below.

Double Coset Algebras

For a group G and a subgroup H of G, the polygroup of all double cosets of H in G ([12],[6]) is denoted by G//H. The class of all polygroups isomorphic to such systems is denoted DBCOSET(Group). Actually, the same construction works for any class of hypergroups, not just groups. Thus, DBCOSET can be regarded as an operation on classes of hypergroups.

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Color Algebras

Let C be a nonempty set (think of the elements of C as colors), $0 \in C$, and $^{-1}$ an involution of C that fixes 0, i.e., $0^{-1} = 0$. A ***C–color scheme*** is a relational system $V = \langle V, R_a \rangle_{a \in C}$ where each R_a is a binary relation on V that satisfies

(1) $\{R_a : a \in C\}$ is a partition of V^2 and $R_0 = \{(x,x) : x \in V\}$

(2) $R_{a^{-1}} = R_a^{\smile}$ (the converse of R_a), and

(3) for every $a,b,c \in C$, $R_c \cap (R_a | R_b) \neq \emptyset$ implies $R_c \subseteq R_a | R_b$.

A more general scheme that satisfies only (1) and (2) is called a chromatic geometry (or rainbow) in Ashbacher [1].

The ***color algebra*** of a C–color scheme $V = \langle V, R_a \rangle_{a \in C}$ is the polygroup $M_V = \langle C, *, 0 \rangle$ where for $a,b \in C$,

$$a*b = \{ c \in C : R_c \subseteq R_a | R_b \}.$$

Let CHROM denote the class of all polygroups isomorphic to a color algebra M_V for some scheme V.

Association Schemes

An ***association scheme*** (Bannai and Ito [3]) is a color scheme $\langle V, R_a \rangle_{a \in C}$ that satisfies

(4) $p^c_{ab} = |\{ z \in V : (x,z) \in R_a \text{ and } (z,y) \in R_b \}|$ is independent of $(x,y) \in R_c$ for every $a,b,c \in C$ and $x,y \in V$.

Association schemes are necessarily finite systems. Let ASSN denote the class of all polygroups isomorphic to the color algebra of an association scheme. These color algebras are closely related to the notion of a Table Algebra defined in Arad and Blau [2].

Double Quotients

The formation of the polygroup $G//H$ is a type of quotient construction. This generalizes as follows: A ***conjugacy relation*** on a polygroup M is an equivalence relation θ such that for all $x,y,z,x' \in M$

(5) $x\theta y$ implies $x^{-1}\theta y^{-1}$, and

(6) $z'\theta z \in x \cdot y$ implies $z' \in x' \cdot y'$ for some $x'\theta x$ and $y'\theta y$.

A conjugacy relation on a polygroup M induces a (***double***) ***quotient*** polygroup $M//\theta = \langle \{\theta x : x \in M\}, *, \theta e \rangle$ where $\theta x = \{ y \in M : y \theta x \}$ and

$$\theta x * \theta y = \{ \theta z : z \in (\theta x)(\theta y) \}.$$

The construction of $M//\theta$ not only generalizes that of a double coset algebra but also the construction that makes a hypergroup from the conjugacy classes of a group. Let $Q^2(K)$ denote the class of all systems isomorphic to algebras $M//\theta$ for $M \in K$ and θ a conjugacy relation on M. A conjugacy relation θ that satisfies

$$x\theta e \implies x = e$$

(recall that e is the unique scalar identity) is called a ***special*** conjugacy relation and the class of corresponding special quotients of systems in K is denoted $Q^2_S(K)$.

Problem 1. Give characterizations of the classes CHROM, ASSN, Q^2(Group), and DBCOSET(Group).

It is known that Group $\subset$ DBCOSET(Group) $\subseteq$ Q^2(Group) $\subset$ CHROM and that Fin$\cap Q^2$(Group) $\subseteq$ Q^2(ASSN) $\subseteq$ Fin$\cap$CHROM where Fin denotes the class of finite systems.

Problem 2. Can any of the inclusions $\subseteq$ above be replaced by equality?

Versions of the problems above also need to be investigated for commutative polygroups ($x \cdot y = y \cdot x$) and symmetric polygroups ($x^{-1} = x$ for all x). Also,

Problem 3. Find a reasonable notion of cyclic polygroup and Abelian polygroup which extend results from group theory.

There have been several attempt to formulate a suitable notion of Abelian association scheme. Ferguson and Turull [13] define a commutative association scheme to be Abelian if

$$p^0_{ii^{-1}} = 1$$

for all $i \in C$. Unfortunately, the color algebras associated with their notion are just ordinary Abelian groups. Arad and Blau [2] have generalized the notion of Abelian group to Table Algebras by abstracting the idea that a group is Abelian exactly when all of its conjugacy classes are trivial. A third approach would be to regard a cyclic polygroup as one that is the color scheme of a P–polynomial association scheme (cf., [3], Chapter 3). In such an association scheme the adjacency matrix of the i–th relation R_i is obtained from the adjacency matrix of the first relation R_1 by a polynomial of degree i. Since P–polynomial schemes correspond to distance regular graphs $\langle V,R_1\rangle$, there is a hope that the corresponding color algebras can be completely classified.

2. EXTENSIONS AND DECOMPOSITIONS

It is useful to decompose systems into "simpler" ones. There are several product operations available for decomposing hypergroups. First, there is the standard direct product of hypergroups. The classes of all polygroups, CHROM, Q^2(Group), and DBCOSET(Group) are closed under direct products and ASSN is closed under the formation of finite direct products(cf., Comer [8], Ferguson and Turull [13]).

A second product notion is that of the wreath product A[B] of a polygroup A by a polygroup B introduced in [7]. (Also, see Ashbacher [1] for an extension to chromatic geometries.) In [11] de Salvo introduced a family of extensions, called (H,G)–hypergroups, of a hypergroup H by a group G. De Salvo's construction still works when the group G is replaced by a polygroup. Unfortunately, none of the (H,G)–hypergroups that are produced when the construction is applied to polygroups H and G are polygroups except the wreath product H[G]. The classes of all polygroups, CHROM, Q^2(Group), and DBCOSET(Group) are all closed under wreath products [8].

Each construction mentioned above gives an ***extension*** P ***of*** A ***by*** B in the sense that A is a subsystem of P and that $P//A \cong B$.

Problem 4. Determine *all* polygroup extensions P of a polygroup A by a polygroup B. In particular, determine all extensions when A and B are restricted to one of the special classes: Group, **DBCOSET**(Group), ASSN, or CHROM.

A desirable description should extend the one that is known when A and B are both groups (Hall [16], Chapter 15). In this regard, the work of Sureau [21] and Vougiouklis [22] is relevant. Sureau studied extensions of a hypergroup A by a group B that are obtained as a semidirect product $A \times_\tau B$ relative to a group homomorphism $\tau: B \longrightarrow Aut(A)$. Using the fundamental equivalence relation β^*_B on a hypergroup B, Vougiouklis introduced a semidirect product $A \times_\tau B$ of two hypergroups A and B relative to a group homomorphism $\tau\colon B/\beta^*_B \longrightarrow Aut(A)$. It should be observed that the notion of extension used in Problem 4 depends on the double coset construction, not the usual coset construction. Of course, the two notions coincide when applied to a normal subsystem.

Suppose EXT(A,B) denotes the class of all isomorphism types of extensions of A by B (cf., [21]).

Problem 5. Does EXT(A,B) have any type of structure?

Problem 6. Develop a reasonable decomposition theory for special classes of polygroups where the indecomposable factors cannot be decomposed by direct products, wreath products, or semidirect products.

The work by Ashbacher [1] on symmetric edge transitive chromatic geometries may provide a clue.

3. d-VARIETIES

For a class K of hypergroups, let S(K) denote the class of all hypergroups isomorphic to a subhypergroup of a member of K and let P(K) denote the class of all hypergroups isomorphic to a direct product of members of K. A class K of hypergroups is a ***d–variety*** if S(K) = K, Q^2(K) = K, and P(K) = K. The class Q^2SP(K) is the smallest d–variety that contains a class K. The collection of all d–varieties contained in a given d–variety forms a complete lattice $\mathscr{L}_d$. Many interesting classes of hypergroups are d–varieties, eg., the class of all polygroups, CHROM, and $Q^2(\mathrm{Group}) = Q^2SP(\mathrm{Group})$.

Problem 7. What is the structure of the lattice $\mathscr{L}_d$?

Let $T = \langle\{0,1\},\vee^*\rangle$ where $\vee^*$ is the operation defined by the table

$\vee^*$	0	1
0	0	1
1	1	0,1

It is easy to show that $Q^2SP(T)$ is the only atom of $\mathscr{L}_d$. What are the covers of this d–variety? Also, $\mathscr{L}_d$ contains a countable antichain, namely, the d–varieties $Q^2SP(\mathbb{Z}_p)$ where p is a prime. Do these d–varieties cover $Q^2SP(T)$?

Problem 8. Find a syntactic notion of "identity" corresponding to the notion of d–variety, i.e., a notion of "identity" whose equational classes are exactly the d–varieties.

It is likely that the notion of "identity" will be a type of Horn formulae (cf., Schweigert [20]). The "identities" should be expressed in a functional language and properties such as the associative law, expressed naturally as $x\cdot(y\cdot z) = (x\cdot y)\cdot z$, should be "identities".

4. CONJUGACY LATTICES

The notion of a conjugacy relation arises naturally in the study of special polygroups. Special conjugacy relations are exactly the kernels of morphisms resulting from the dual equivalence between polygroups and complete atomic integral relation algebras ([6]). When the notion of homomorphism between commutative association schemes ([13]) is lifted to the corresponding color algebras, the kernels of the resulting maps are conjugacy relations. The collection of all conjugacy relations on a polygroup M forms a complete lattice, denoted Conj(M), and the class of all special conjugacy relations on M, denoted $\mathrm{Conj}_s(M)$, is a principal ideal (hence sublattice) of Conj(M).

Problem 9(a). Does the lattice Conj(G) of a finite group G determine G?
(b). Does $\mathrm{Conj}_s(G)$ determine G when G is a finite group with $|G| \geq 6$?

The reason for the restriction in Problem 9(b) is that $\mathrm{Conj}_s(\mathbb{Z}_4) \cong \mathrm{Conj}_s(\mathbb{Z}_5)$. Problem 9 is also open for infinite groups. For finite Abelian groups the problem reduces to the groups $\mathbb{Z}_p$ for p prime (Comer [10]).

Problem 10. Compute the number n(p) of elements in $\mathrm{Conj_S}(\mathbb{Z}_p)$ that are simultaneously atoms and dual atoms of the lattice. Does this number determine p if p is a prime ≥ 11?

It is known that $n(5) = 1$, $n(7) = 2$, $n(11) = 11$, $n(13) = 24$, and $n(p) \geq p-1$ for all primes $p \geq 11$.

5. AXIOMATIZABILITY PROBLEMS

In [14] Haddad and Sureau showed that the class of all D–hypergroups is not an elementary class (in the language of a 3–place relation) and that the first–order theory of D–hypergroups is not finitely axiomatizable. In [15] cogroups were constructed that are not of Utumi type, i.e., they are not obtainable from a D–hypergroup H by defining a new operation $*$ on H as $x*y = x\cdot\theta y$ where θ is an Utumi partition on H. The class of cogroups of Utumi type can be characterized using multipliers ([15]).

Problem 11(a). Find a reasonable set of axioms (necessarily infinite) for the elementary theory of D–hypergroups.
(b). Is the class of cogroups of Utumi type an elementary class? Is its elementary theory finitely axiomatizable over the theory of D–hypergroups?

Three techniques are known for constructing cogroups: the D–hypergroup construction, the Utumi construction, and the Haddad/Sureau construction. Is this enough? Two forms of this question are given below.

Problem 12(a). Find a collection of constructions which allows one to determine all (finite) cogroups from groups.
(b). Is every weak cogroup equivalent to a cogroup of Utumi type?

To understand (b) we need to recall some notation from [9]. On a weak cogroup H (also, called a hypergroup of type C in [14]) an equivalence relation $\approx$ can be defined by

$$x \approx y \iff x\cdot e = y\cdot e$$

(e is the left scalar identity of H). The collection of all $\approx$–equivalence classes $H/\approx$ is a polygroup with the natural operations. Every such polygroup is in CHROM and every polygroup in Q^2(Group) has this form (cf., [9]). Two weak cogroups H and K are called ***equivalent*** if $H/\approx$ is isomorphic to $K/\approx$. Problem 12(b) ask for a characterization of cogroups up to (polygroup) equivalence.

Problem 13. Is DBCOSET(Group) an elementary class? Is every finite model of its theory isomorphic to a double coset algebra?

If F is a field and $G \triangleleft F^*$ (= the multiplicative group of a field F), the factor hyperring F/G gives a hyperfield called a ***quotient hyperfield*** (Krasner [18]).

Massouros [19] showed there exist hyperfields not isomorphic to quotient hyperfields.

Problem 14. Is the class of quotient hyperfields an elementary class?

6. ORDERED HYPER–STRUCTURES

The classical development of Sentential Logic is based on the three operations $\wedge$, $\vee$, and $\neg$ whose tables are as follows:

$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	1
1	1	1

$\neg$	
0	1
1	0

What if the $\vee$ operation is replaced by $\vee^*$ defined in Section 3?

Problem 15. Develop a Sentential hyper–logic based on the operations $\wedge$, $\vee^*$, and $\neg$. That is, axiomatize the logic and investigate properties such as the deduction theorem and functional completeness.

Since Boolean algebras provide an algebraic analog of Sentential logic, Boolean hyper–algebras, studied by Konstantinidou and Mittas [17], should be the algebraic analog to Sentential hyper–logic. Does a form of the Stone representation result hold?[2]

What about algebraic versions of non–classical logics? For example, a ***Heyting algebra*** $<L,\vee,\wedge,0,1,\rightarrow>$ is a bounded lattice $<L,\vee,\wedge,0,1>$ such that for all $a,b \in L$ $\{ x \in L : a \wedge x \leq b \}$ contains a greatest element $a \rightarrow b$, called the relative pseudo–complement of a in b ([4], Chapter 2,§11). The existence of $a \rightarrow b$ for every $a,b \in L$ implies that L is distributive. What if we weaken the *uniqueness* of the relative pseudo–complement? Let us say that a system $<L,\vee,\wedge,0,1,\rightarrow>$ is a ***Heyting hyper–algebra*** if $<L,\vee,\wedge,0,1>$ is a bounded lattice and for all $a,b \in L$ the set $\{ x \in L : a \wedge x \leq b \}$ contains maximal elements, the set of which is denoted by $a \rightarrow b$.

Problem 16. How much of the theory of Heyting algebras carries over to Heyting hyper–algebras and is there an analogous version of "Intuitionistic hyper–logic"?

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[2]Recently, Rita Procesi–Ciampi and Rosaria Rota answered this question affirmatively.

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