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# AN ALGEBRAIC APPROACH TO THE APPROXIMATION OF INFORMATION

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**Abstract.** This paper is based on the notion of an information system $<U,\Omega,V,f>$ in the sense of Pawlak. Every set of knowledge $P \subseteq \Omega$ determines a closure operator on $U$. The class of Boolean algebras with added operations determined by all sets of knowledge are axiomatixed. As a consequence of the representation theorem information systems can be constructed that have a prescribed lattice of functional dependencies.

## 1. Introduction

This paper deals with the notion of an information system $S = <U,\Omega,V,f>$ in the sense of Pawlak [6]. These information systems have been studied under various names: databases, knowledge representation systems, decision tables, and learning systems ([4], [7], [8], [10]). In the approach taken by Pawlak, a subset $P$ of $\Omega$ is called a set of knowledge and determines an approximation space $<U,\theta_P>$ and a closure operator $\bar{P}$ on $U$. In the methodology of rough concepts, $\bar{P}X$ denotes the $P$–upper approximation of a concept $X \subseteq U$. The closure algebras $<\mathfrak{Sb}\,U,\bar{P}>$, where $\mathfrak{Sb}\,U$ is the Boolean algebra of all subsets of $U$, can be characterized as complete atomic cylindric algebras of dimension 1 (Proposition 14).

Often one is interested in relationships between various sets of knowledge. An algebraic framework for studying this situation is developed in this paper. Every information system $S$ determines a Boolean algebra with unary operators $<\mathfrak{Sb}\,U,\bar{P}>_{P\subseteq\Omega}$ which is called a *knowledge approximation algebra of type* $\Omega$ *derived from* $S$. We propose a (non–elementary) set of axioms for the class of all such algebras of a fixed type and show that the axioms have the intended models (Theorem 11). Finally, in Section 4 it is shown that the first–order theory of knowledge approximation algebras of type $\Omega$, as well as the theory of its finite models, is undecidable whenever $|\Omega| \geq 2$.

Throughout the paper we assume that $\Omega$ is a finite set. We use [2] as our basic reference for notation; in particular, $SbX$ denotes the collection of all subsets of $X$

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and $\mathfrak{S}bX$ denotes the Boolean algebra with universe $SbX$.

## 2. Basic Definitions and Elementary Properties

An *information system* is a 4–tuple $S = <X,\Omega,V,f>$ where $X$ is a set, $\Omega$ is a finite set, $V$ is a function with $\mathrm{Dom}\,V = \Omega$ and $f.X \longrightarrow \prod_{a\in\Omega} V_a$. For each $P \subseteq \Omega$, define a relation $\theta_P$ for $x,y \in X$ by

$$x\theta_P y \iff \forall a\in P\ (fx)_a = (fy)_a.$$

Clearly $\theta_P$ is an equivalence relation on $X$. The pair $(X, \theta_P)$ is called an *approximation space* for knowledge $P$ and the $\theta_P$–classes, i.e., the subsets $\theta_P x = \{ y : x\theta_P y \}$ are called $P$–*elementary categories* or concepts *indiscernible* according to knowledge $P$. A set $A \subseteq X$ is *definable in knowledge* $P$ if $A$ is a union of $\theta_P$–classes, i.e., $A = \bigcup \{ \theta_P x : x\in A \}$.

Associated with an approximation space $(X,\theta_P)$ there is a closure operator $\bar{P}$ and an interior operator $\underline{P}$ on $X$. Define $\bar{P} : SbX \longrightarrow SbX$ and $\underline{P}: SbX \longrightarrow SbX$ by

$$\bar{P}(A) = \bigcup \{ \theta_P x : x\in A \} \quad \text{for } A \subseteq X \text{ and}$$
$$\underline{P}(A) = \bigcup \{\theta_P x : \theta_P x \subseteq A \} \quad \text{for } A \subseteq X.$$

Pawlak ([7],[8]) calls $\bar{P}(A)$ the $P$–*upper approximation* of $A$ and $\underline{P}(A)$ the $P$–*lower approximation* of $A$. Note that the subsets of $X$ that are definable in $P$ are the fixed points of $\bar{P}$ (or the $\bar{P}$–closed subsets).

The structure $\mathfrak{B}_S = <SbX,\cup,\cap,\sim,\emptyset,X,\bar{P}>_{P\subseteq\Omega}$ (or $<\mathfrak{S}bX,\bar{P}>_{P\subseteq\Omega}$ for short) is called the *knowledge approximation algebra of type* $\Omega$ *derived from the information system* $S$. The reduct $\mathfrak{Rd}_P\mathfrak{B}_S = <SbX,\cup,\cap,\sim,\emptyset,X,\bar{P}>$ is called the (upper) *approximation closure algebra of* $P$.

The next definition presents axioms for an abstract knowledge approximation algebra of type $\Omega$. The idea is to abstract the properties of the closure operator $\bar{P}$ as an operator $\kappa_P$.

Definition. A structure $\mathfrak{B} = <\mathscr{B},\kappa_P>_{P\subseteq\Omega}$ is a *knowledge approximation algebra of type* $\Omega$ (recall that $\Omega$ is finite) if $\kappa_P \in B^B$ for each $P \subseteq \Omega$ and the following axioms hold for all $x,y\in B$ and $P,Q \subseteq \Omega$:

($A_0$) $\mathscr{B} = <B,+,\cdot,-,0,1>$ is a complete atomic Boolean algebra,

($A_1$) $\kappa_P 0 = 0$,

($A_2$) $\kappa_P x \geq x$,

($A_3$) $\kappa_P(x\cdot\kappa_P y) = \kappa_P x\cdot\kappa_P y$,

($A_4$) $x \neq 0$ implies $\kappa_\emptyset x = 1$,

($A_5$) $\kappa_{P\cup Q}x = (\kappa_P x)\cdot(\kappa_Q x)$ if $x$ is an atom of $\mathscr{B}$

$\mathfrak{B}$ is called *reduced* if $\kappa_\Omega x = x$ for all $x \in B$. We denote the class of all knowledge approximation algebras of type $\Omega$ by $\mathbf{KA}_\Omega$ and refer to a member of this class as a $\mathrm{KA}_\Omega$.

Observe that if $\mathfrak{B} = <\mathscr{B},\kappa_P>_{P\subseteq\Omega}$ is a $\mathrm{KA}_\Omega$, then axioms ($A_0$)–($A_3$) show that $<\mathscr{B},\kappa_P>$ is a $\mathrm{CA}_1$ for each $P \subseteq \Omega$, i.e., each $\kappa_P$ is a cylindrification in the sense of [2]. The *dual operation* $\kappa_P^\partial$ associated with a cylindrification $\kappa_P$ (cf., 1.4.1 of [2]) is defined by $\kappa_P^\partial x = -\kappa_P(-x)$ for all $x \in B$.

Proposition 1. If $S = <X,\Omega,V,f>$ is an information system, the knowledge approximation algebra $\mathfrak{B}_S$ of type $\Omega$ derived from $S$ is a $\mathrm{KA}_\Omega$. In particular, every reduct $\mathfrak{Rd}_P\mathfrak{B}_S = <SbX,\cup,\cap,\sim,\emptyset,X,\bar{P}>$ is a $\mathrm{CA}_1$.

Proof. Clearly ($A_0$),($A_1$), and ($A_2$) hold.

($A_3$) $\bar{P}(A\cap\bar{P}B) = \bar{P}A\cap\bar{P}B$ for $A,B \subseteq X$.

Suppose $x \in \bar{P}A\cap\bar{P}B$. Then $\theta_P\cap A \neq \emptyset$ and $\theta_P\cap B \neq \emptyset$. Since $\bar{P}B$ is a union of $\theta_P$–classes and $x \in \bar{P}B$, $\theta_P x \subseteq \bar{P}B$. Thus, $A\cap\bar{P}B\cap\theta_P x = A\cap\theta_P x \neq \emptyset$. Hence $x \in \theta_P x \subseteq \bar{P}(A\cap\bar{P}B)$; so the inclusion $\supseteq$ holds. Now, suppose $x \in \bar{P}(A\cap\bar{P}B)$. Then $A\cap\bar{P}B\cap\theta_P x \neq \emptyset$; so $A\cap\theta_P x \neq \emptyset$ (therefore $x \in \bar{P}A$ ) and $\bar{P}B\cap\theta_P x \neq \emptyset$ (which implies $x \in \theta_P x \subseteq \bar{P}B$ because $\bar{P}B$ is a union of $\theta_P$–classes). Hence the inclusion $\subseteq$ holds.

($A_4$) $\emptyset \neq A \subseteq X \Longrightarrow \bar{\emptyset}A = X$

If $x \in A$ and $y \in X$, then $x\theta_\emptyset y$ holds vacuously. So $\bar{\emptyset}\{x\} = X$ and ($A_4$) follows.

($A_5$) $\overline{P\cup Q}\{x\} = \bar{P}\{x\}\cap\bar{Q}\{x\}$ for all $x \in X$.

If either $P$ or $Q$ is empty, the result follows from ($A_4$). Assume $P,Q \neq \emptyset$. Suppose $y \in \bar{P}\{x\}\cap\bar{Q}\{x\}$. Then $x\theta_P y$ and $x\theta_Q y$. So, if $a \in \theta_{P\cup Q}$, then $(fx)_a = (fy)_a$ because either $a \in P$ or $a \in Q$. Therefore, $x\theta_{P\cup Q}y$, i.e., $y \in \overline{P\cup Q}\{x\}$. Thus, the inclusion $\supseteq$ holds. Now suppose that $y \in \overline{P\cup Q}\{x\}$ so $x\theta_{P\cup Q}y$. Since $(fx)_a = (fy)_a$ for all $a \in P\cup Q$, $fx$ and $fy$ agree on all values in $P$ and on all values in $Q$. Thus $x\theta_P y$ and $x\theta_Q y$. Hence the inclusion $\subseteq$ holds.

Note that the algebra $\mathfrak{B}_S$ derived from an information system $S = <X,\Omega,V,f>$ is reduced if and only if $f$ is one–one.

COROLLARY 2. The dual of $\bar{P}$ in $\mathfrak{B}_S$ is $\underline{P}$, i.e., $\kappa_P^\partial = \underline{P}$ when $\kappa_P = \bar{P}$.

PROOF. Since $\kappa_P^\partial x = -\kappa_P(-x)$ it suffices to show $-\bar{P}(-A) = \underline{P}A$ for $A \subseteq X$. This follows since both $-\bar{P}(-A)$ and $\underline{P}A$ are unions of $\theta_P$–classes and $\theta_P x \subseteq A$ iff $\theta_P x \cap -A = \emptyset$ iff $\theta_P x \subseteq -\bar{P}(-A)$.

The result below summarizes properties from Sections 1.2 and 1.4 of [2] which hold because $<B,\kappa_P>$ is a $CA_1$ for each $P \subseteq \Omega$. Thus, the properties follow from axioms $(A_0)$–$(A_3)$ only. It is well known that these properties hold for closure operators $\bar{P}$ and $\underline{P}$ associated with approximation spaces ([7], [8]).

PROPOSITION 3. (i) $\kappa_P^\partial x \leq x \leq \kappa_P x$.

(ii) $\kappa_P^\partial 0 = \kappa_P 0 = 0$ and $\kappa_P^\partial 1 = \kappa_P 1 = 1$.

(iii) $x \leq y \implies \kappa_P x \leq \kappa_P y$ and $\kappa_P^\partial x \leq \kappa_P^\partial y$.

(iv) $\kappa_P(x+y) = \kappa_P x + \kappa_P y$ and $\kappa_P^\partial(x\cdot y) = (\kappa_P^\partial x)\cdot(\kappa_P^\partial y)$.

(v) If $\Sigma_i z_i$ exist, then $\Sigma_i \kappa_P z_i$ exist and $\kappa_P(\Sigma_i z_i) = \Sigma_i \kappa_P z_i$.

(vi) $\kappa_P \kappa_P x = \kappa_P^\partial \kappa_P x = \kappa_P x$.

(vii) $\kappa_P^\partial \kappa_P^\partial x = \kappa_P \kappa_P^\partial x = \kappa_P^\partial x$.

(viii) If $\Pi_i \kappa_P z_i$ exist, then $\kappa_P(\Pi_i \kappa_P z_i) = \Pi_i \kappa_P z_i$ ; in particular, $\kappa_P(\kappa_P x \cdot \kappa_P y) = \kappa_P x \cdot \kappa_P y$.

The following lemma describes natural relationships between approximation operators which will be used in Section 3.

LEMMA 4. (i) $\kappa_P x \leq \kappa_Q x$ for all $x \in B$ whenever $Q \subseteq P \subseteq \Omega$.

(ii) $x \leq \kappa_P y \implies \kappa_P x \leq \kappa_P y$.

PROOF. (i) By additivity 3(v), it suffices to consider $x \in \mathrm{At}\mathfrak{B}$ and $Q \subseteq P$. Then, by $(A_5)$,

$$\kappa_P x = \kappa_{P \cup Q} x = (\kappa_P x)\cdot(\kappa_Q x) \leq \kappa_Q x.$$

(ii) follows from 3(iii) and 3(vi).

### 3. Representation

The goal of this section is to show that every $KA_\Omega$ can be obtained from an information system, i.e., axioms $(A_0)$–$(A_5)$ have the intended models. We then examine the relationship between approximation closure algebras and $CA_1$'s.

The first step is to understand the structure of the class of approximation operators of a $KA_\Omega$, that is, we want to characterize the class $\{ \kappa_P : P \subseteq \Omega \}$ of closure operators of a $KA_\Omega$ $\mathfrak{B}$. By additivity 3(v) and the fact $\mathfrak{B}$ is complete and atomic, each $\kappa_P$ is determined by its values on $\mathrm{At}\mathfrak{B}$. Thus $\kappa_P$ may be viewed as a member of $B^{\mathrm{At}\mathfrak{B}}$. More precisely, let $\bar{\kappa}_P = \kappa_P | \mathrm{At}\mathfrak{B}$, the restriction of $\kappa_P$ to the atoms. By $(A_5)$, $\{ \bar{\kappa}_P : P \subseteq \Omega \}$ is a meet–subsemilattice of $<B^{\mathrm{At}\mathfrak{B}}, \wedge>$ where $\wedge$ is defined pointwise in $B^{\mathrm{At}\mathfrak{B}}$, i.e., $(f \wedge g)(x) = fx \cdot gx$ for $x \in \mathrm{At}\mathfrak{B}$,. In fact, $(A_5)$ shows that the map that sends $P \longmapsto \bar{\kappa}_P$ is a semilattice morphism $<\mathfrak{Sb}\Omega, \cup> \longrightarrow <B^{\mathrm{At}\mathfrak{B}}, \wedge>$. Instead of using the meet–subsemilattice of $<B^{\mathrm{At}\mathfrak{B}}, \wedge>$ given above, we use an isomorphic subsemilattice of the partition lattice $\Pi(\mathrm{At}\mathfrak{B})$ whose elements consist of the partitions that are kernels of the $\bar{\kappa}_P$'s.

Definition 5. For a $KA_\Omega$ $\mathfrak{B} = <B, \kappa_P>_{P \subseteq \Omega}$ the *partition semilattice* of $\mathfrak{B}$, denoted by $L_{\mathfrak{B}}$, is the structure $<\{T_P : P \subseteq \Omega\}, \cap, U^2>$ where $U = \mathrm{At}\mathfrak{B}$ and, for $P \subseteq \Omega$, $a, b \in U$

$$aT_Pb \iff \kappa_P a = \kappa_P b$$

(or equivalently, $aT_Pb \iff a \leq \kappa_P b$).

Lemma 6. (i) $T_P$ is an equivalence relation on $U$ for all $P \subseteq \Omega$.
(ii) $T_\emptyset = U^2$.
(iii) $T_P \cap T_Q = T_{P \cup Q}$ for all $P, Q \subseteq \Omega$.
(iv) $L_{\mathfrak{B}}$ is a meet–subsemilattice of $\Pi(U)$.
(v) The map $P \longmapsto T_P$ is a morphism of $<Sb\Omega, \cup, \emptyset>$ onto $L_{\mathfrak{B}}$.

Proof. (i) obvious and (ii) follows from $(A_4)$. Parts (iv) and (v) are immediate from (i) – (iii).
(iii). Suppose $(a,b) \in T_P \cap T_Q$. Then $\kappa_P a = \kappa_P b$ and $\kappa_Q a = \kappa_Q b$. By $(A_3)$,

$$\kappa_{P \cup Q} a = \kappa_P a \cdot \kappa_Q a = \kappa_P b \cdot \kappa_Q b = \kappa_{P \cup Q} b$$

since $a, b \in \mathrm{At}\mathfrak{B}$. Therefore, the inclusion $\subseteq$ holds. Now, suppose $\kappa_{P \cup Q} a = \kappa_{P \cup Q} b$. Then $b \leq \kappa_{P \cup Q} b = \kappa_{P \cup Q} a \leq \kappa_P a$ so $\kappa_P b \leq \kappa_P a$ by 4(ii). Similarly $\kappa_P a \leq \kappa_P b$ so $\kappa_P a = \kappa_P b$ and $(a,b) \in T_P$. Likewise $(a,b) \in T_Q$; so the inclusion $\supseteq$ holds.

$\Leftrightarrow\ T_{\{a\}}x = T_{\{a\}}y$

$\Leftrightarrow\ (fx)_a = v_{a,b} = (fy)_a$ where $b = T_{\{a\}}x$ (by Def. 9)

$\Leftrightarrow\ y\theta_{\{a\}}x$

$\Leftrightarrow\ y \in \theta_{\{a\}}x = \kappa^{\mathfrak{B}_S}_{\{a\}}(x) = \kappa^{\mathfrak{B}_S}_{\{a\}}(gx).$

Hence, (2) holds. Now, for $x \in \mathrm{At}\mathfrak{B}$ and $P \neq \emptyset$, (1) follows from (2) and ($A_5$):

$$g(\kappa_P x) = g(\prod_{a \in P} \kappa_{\{a\}} x) = \prod_{a \in P} g(\kappa_{\{a\}} x) = \prod_{a \in P} \kappa_{\{a\}}(gx) = \kappa_P(gx).$$

Observe that, by ($A_4$), (1) obviously holds if $P = \emptyset$. Hence, $g$ is an isomorphism of $\mathfrak{B}$ onto $\mathfrak{B}_{S(L)}$ as required.

The next few observations deal with consequences of Theorem 11.

REMARK 12. Theorem 11 shows that axioms ($A_0$)–($A_5$) completely characterize the class of knowledge approximation algebras derived from information systems. Axioms ($A_1$)–($A_3$) are equations while ($A_4$) and ($A_5$) are not. The lemma below shows the class of all $KA_\Omega$'s is not equational.

LEMMA 13. Every member of $S(KA_\Omega)$ is a simple (universal) algebra.

PROOF. It suffices to show, from axioms ($A_0$)–($A_4$), that any congruence relation on a $KA_\Omega$ $\mathfrak{B}$ that is not the identity relation is the universal relation. If $\theta$ is a congruence on $\mathfrak{B}$ and $\theta \neq \mathrm{Id}$, then $\theta$ is a Boolean congruence; so there exist an atom $x \in B$, $x\theta 0$. Then $1 = (\kappa_\emptyset x)\theta(\kappa_\emptyset 0) = 0$ by ($A_4$). Hence, $\theta$ is the universal congruence.

As noted in Proposition 1 every approximation closure algebra $<\mathfrak{Sb}\,U,\bar{P}>$ associated with an information system is a complete atomic $CA_1$. Below we see that every complete atomic $CA_1$ has such a representation.

PROPOSITION 14. Let $P_0 \subseteq \Omega$ with $P_0 \neq \emptyset$. Then

(i) Every complete atomic $CA_1$ is isomorphic to an approximation closure algebra $\mathfrak{Rd}_{P_0}\mathfrak{B}_S = <\mathfrak{Sb}\,U,\bar{P}_0>$ for some information system $S$.

(ii) Every $CA_1$ is embeddable in an approximation closure algebra $\mathfrak{Rd}_{P_0}\mathfrak{B}_S$ for some $S$.

PROOF. (i) Given a complete atomic $CA_1$ $\mathfrak{A} = <A,c_0>$ we define a system $\mathfrak{B} = <B,\kappa_P>_{P\subseteq\Omega}$ by letting the Boolean algebra $B = A$, $\kappa_P = c_0$ for all $P \neq \emptyset$, and, for all $x \in A$, $\kappa_\emptyset x = 1$ if $x \neq 0$ and equal 0 otherwise. It is clear that $\mathfrak{B}$ is a $KA_\Omega$ so, by

Theorem 11, $\mathfrak{B} \cong \mathfrak{B}_S$ for some information system $S$. Thus, $\mathfrak{A} = \mathfrak{Rd}_{P_0}\mathfrak{B} \cong \mathfrak{Rd}_{P_0}\mathfrak{B}_S$ as desired.
(ii) follows from (i) by the fact that every $CA_1$ is embeddable in a complete atomic one (cf., 2.7.20 of [2]).

The final result of this section uses Proposition 8 and Theorem 11 to create an information system with a prescribed relation of functional dependencies.

THEOREM 15. Every finite lattice is isomorphic to the partition semilattice of some $KA_\Omega$. Moreover, this algebra may be chosen as the algebra derived from the information system $S(L')$ where $L'$ is a representation of the lattice as a meet semilattice of partitions.

PROOF. Given a finite lattice $L$ first observe that $L$ is isomorphic to a meet–semilattice $L'$ of partitions. This is immediate from the Pudlák and Tůma solution [9] of Whitman's problem; however, we provide a simple direct construction. For $x \in L$ let $h(x)$ be the equivalence relation on $L$ defined for $a,b \in L$ by

$$a h(x) b \quad \Leftrightarrow \quad a = b \text{ or } a,b \leq x.$$

Then $h : L \longrightarrow \Pi(L)$ is a meet–semilattice isomorphism of $L$ onto $L' = \{ h(x) : x \in L \} \subseteq \Pi(L)$. Now, let $\Omega = L$ and, for $P \subseteq \Omega$, set $T_P = \bigcap_{x \in P} h(x)$. Note that $T_P = h(\wedge P)$ since $h$ is a meet–isomorphism of $L$ onto $L'$. Then $L' = \{ T_P : P \subseteq \Omega\}$ and $<L', \cap, L^2>$ is a partition semilattice. Since the map that sends $P \subseteq \Omega$ to $h(\wedge P)$ is a semilattice morphism from $<Sb\Omega, \cup, \emptyset>$ onto $L'$ it follows from Proposition 8 that $L'$ is isomorphic to the partition semilattice of some $KA_\Omega$ $\mathfrak{B}$. By applying Theorem 11 to $\mathfrak{B}$ with $L'$ as its partition semilattice we see that the information system $S(L')$ has the property that $\mathfrak{B} \cong \mathfrak{B}_{S(L')}$.

## 4. Decision Problems

The goal of this section is to settle the decision problems for the first–order theory of $KA_\Omega$ for every finite $\Omega$. The answers are closely related to the ones obtained for cylindrification algebras in [1] but the details differ. We will use certain basic facts about finitely inseparable theories which can be found in either [1] or Monk [5] (results 15.7, 15.16, and 16.56).

Let **Eq** denote the theory of two equivalence relations, i.e., the models of **Eq** are relational structures $<X,R,S>$ where $R$ and $S$ are equivalence relations on $X$. The theory **Eq** is finitely inseparable by 16.56 of Monk [5]. To show that a theory **T**

REMARK 7. (i) $L_{\mathfrak{B}}$ is actually a lattice, but not necessarily a sublattice of $\Pi(U)$. The join $\oplus$ is defined by

$$T_P \oplus T_Q = \cap\{ T_R : R \subseteq \Omega \text{ and } T_R \supseteq T_P \text{ and } T_R \supseteq T_Q \}.$$

In Lee [4] the lattice $L_{\mathfrak{B}}$ is called the *relation lattice* of $S$ when $\mathfrak{B}$ is the knowledge approximation algebra induced by an information system (database) $S$. The ordering relation of this lattice expresses the functional dependencies that are valid in $S$.

(ii) We say that a collection of equivalence relations $L = \langle\{T_P : P \subseteq \Omega\},\cap, U^2\rangle$ is a *partition semilattice* if properties 6(i), 6(ii), and 6(iii) hold. Of course, Lemma 6 shows that $L_{\mathfrak{B}}$ is a partition semilattice.

(iii) The collection of relations that form a partition semilattice $L$ is closely related to the notion of a cylindric atom structure (cf., 2.7.40 of [2]). We say that a relational system $\langle U, T_P\rangle_{P\subseteq\Omega}$ is a *knowledge approximation atom structure* if for all $P,Q \subseteq \Omega$

(1) $T_P$ is an equivalence relation on $U$,

(2) $T_\emptyset = U^2$, and

(3) $T_P \cap T_Q = T_{P\cup Q}$

We will use partition semilattices and atom structures interchangeably. The next result characterizes the partition semilattices obtained from $\mathrm{KA}_\Omega$s.

PROPOSITION 8. A meet–subsemilattice of $\Pi(U)$, $\langle L,\cap,U^2\rangle$, is a partition semilattice of some $\mathfrak{B}$ in $\mathrm{KA}_\Omega$ if and only if there is a semilattice morphism of $\langle Sb\Omega,\cup,\emptyset\rangle$ onto $L$.

PROOF. The $\Longrightarrow$ direction follows from 6(v). For $\Longleftarrow$, suppose $L = \{ F_P : P \subseteq \Omega \}$. For each equivalence relation $F_P$ define $F_P^* : SbU \longrightarrow SbU$ by

$$F_P^* X = \bigcup_{x\in X}\{ y\in U : xF_P y \}.$$

Then $\mathfrak{B}' = \langle \mathfrak{S}b\,U, F_P^*\rangle_{P\subseteq\Omega}$ is a $\mathrm{KA}_\Omega$. Clearly, $\mathrm{At}\mathfrak{B}' = \{\{x\} : x\in U\}$. Using the natural correspondence between the atoms of $\mathfrak{B}'$ and the elements of $U$ we obtain a knowledge approximation algebra $\mathfrak{B}$ isomorphic to $\mathfrak{B}'$ such that $\mathrm{At}\mathfrak{B} = U$ and $L_{\mathfrak{B}}$ is $L$.

Another way to state Proposition 8 is to say that a meet–subsemilattice of $\Pi(U)$ is a partition semilattice of a $\mathrm{KA}_\Omega$ if and only if its elements are the relations of a knowledge approximation atom structure.

We now construct an information system from a partition semilattice.

Definition 9. Suppose $L = <\{T_P : P \subseteq \Omega\}, \cap, U^2>$ is a partition subsemilattice of $\Pi(U)$ and $\Omega$ is finite. The structure

$$S(L) = <U, \Omega, V, f>,$$

called the *information system of L*, is defined in the following way. Choose a function $V$ on $\Omega$ such that $|V_a|$ equals the cardinality of the set of $T_{\{a\}}$–blocks for all $a \in \Omega$. We denote the elements of $V_a$ by $\mathrm{v}_{a,b}$ where $b \in U/T_{\{a\}}$ (i.e., $b = T_{\{a\}}x$ for some $x \in U$). Now, define $f: U \longrightarrow \prod_{a \in \Omega} V_a$ by

$$f(x)_a = \mathrm{v}_{a,b}$$

for all $x \in U$, $a \in \Omega$, and $b = T_{\{a\}}x$. Of course, $V_a$ may be infinite if $U$ is infinite.

It is obvious that

Lemma 10. $S(L)$ is an information system.

Theorem 11. If $\mathfrak{B} = <B, \kappa_P>_{P \subseteq \Omega}$ is a $\mathrm{KA}_\Omega^\lambda$ where $\Omega$ is finite, $L = L_{\mathfrak{B}}$ is the partition semilattice of $\mathfrak{B}$, and $S = S(L)$ is the information system of $L$, then $\mathfrak{B} \cong \mathfrak{B}_{S(L)}$, the knowledge approximation algebra of $S(L)$.

Proof. Suppose $L = <\{T_P : P \subseteq \Omega\}, \cap, U^2>$ where $U = \mathrm{At}\mathfrak{B}$ and $S(L) = <U, \Omega, V, f>$ is given in Definition 9. Consider the map $g\colon B \longrightarrow SbU$ defined, for $b \in B$, by

$$g(b) = \{x \in U : x \leq b\}.$$

Since $\mathfrak{B}$ is a complete atomic BA, $g$ is a Boolean isomorphism of $\mathfrak{B}$ onto $\mathfrak{B}_{S(L)}$. It remains to show that

(1) $\quad g(\kappa_P^{\mathfrak{B}} x) = \kappa_P^{\mathfrak{B}_s}(gx)$ for all $x \in B$ and $P \subseteq \Omega$.

Since each $\kappa_P$ and $g$ are completely additive and $\mathfrak{B}$ is atomic, it suffices to verify (1) for $x \in \mathrm{At}\mathfrak{B}$. First, we consider the case where $P = \{a\}$ is an atom of $\mathfrak{S}b\Omega$. We claim that

(2) $\quad g(\kappa_{\{a\}}^{\mathfrak{B}} x) = \kappa_{\{a\}}^{\mathfrak{B}_s}(gx)$ for all $x \in \mathrm{At}\mathfrak{B}$.

Let $x, y \in \mathrm{At}\mathfrak{B}$. Then

$$\begin{aligned} y \in g(\kappa_{\{a\}}^{\mathfrak{B}} x) &\Leftrightarrow y \leq \kappa_{\{a\}}^{\mathfrak{B}} x \\ &\Leftrightarrow y T_{\{a\}} x \text{ (by Def. 5)} \end{aligned}$$

is finitely inseparable, by 15.16 of Monk [5], it suffices to find formulas $\theta v_0$, $\bar{R}v_0v_1$, and $\bar{S}v_0v_1$ in the language of T such that

($M_0$) for every finite model $\mathfrak{A} = <X,R,S>$ of **Eq** there is a finite model $\mathfrak{B}$ of T such that $<\theta^{\mathfrak{B}},\bar{R}^{\mathfrak{B}},\bar{S}^{\mathfrak{B}}> \cong \mathfrak{A}$.

This procedure will be used in the proof of 16(iii) below.

THEOREM 16. (i) The theory of $KA_\Omega$ is decidable if $|\Omega| \leq 1$.
(ii) If $P \subseteq \Omega$ and $P \neq \emptyset$, the theory of $\mathfrak{Rd}_P(KA_\Omega) = \{ \mathfrak{Rd}_P\mathfrak{B} : \mathfrak{B} \in KA_\Omega \}$ is decidable.
(iii) If $|\Omega| \geq 2$, the theory of $KA_\Omega$ is finitely inseparable.

PROOF. (i) There are two cases: $\Omega = \emptyset$ and $\Omega = \{p\}$. $KA_\emptyset$ consist of complete atomic simple $CA_1$'s $<B,\kappa_\emptyset>$. Since $\kappa_\emptyset$ is definable in the theory of BA's, $KA_\emptyset^{?}$ is equivalent to the theory of complete atomic BA's which is decidable. In the case $\Omega = \{p\}$, by 14(i), $KA_\Omega$ consist of algebras $<B,\kappa_\emptyset,\kappa_\Omega>$ where $\kappa_\emptyset$ is Boolean definable and $<B,\kappa_\Omega>$ is a complete atomic $CA_1$. In [1], Section 2, it is shown that the theory of complete atomic $CA_1$'s is the same as the theory of finite $CA_1$'s and that this theory is decidable.
(ii) By 14(i) the theory of $\mathfrak{Rd}_P(KA_\Omega)$ is the theory of complete atomic $CA_1$'s which was shown to be decidable in [1].
(iii) Suppose $|\Omega| \geq 2$ and choose $r,s \in \Omega$ with $r \neq s$. The following formulas give a translation of **Eq** into the language of $KA_\Omega$.

$\theta v_0$: $v_0$ is an atom

$\bar{R}v_0v_1$: $\theta v_0 \wedge \theta v_1 \wedge \kappa_{\{r\}}v_0 = \kappa_{\{r\}}v_1$

$\bar{S}v_0v_1$: $\theta v_0 \wedge \theta v_1 \wedge \kappa_{\{s\}}v_0 = \kappa_{\{s\}}v_1$

If $\mathfrak{B}$ is a $KA_\Omega$, 6(i) shows that $<At\mathfrak{B},\bar{R}^{\mathfrak{B}},\bar{S}^{\mathfrak{B}}>$ is a model of **Eq**. To verify property ($M_0$) suppose $\mathfrak{A} = <X,R,S>$ is a (finite) model of **Eq**. Construct a Knowledge approximation atom structure $<X,T_P>_{P\subseteq\Omega}$ in the following way: let

$$T_{\{r\}} = R,\quad T_{\{s\}} = S,\quad T_\emptyset = X^2,\quad T_{\{i\}} = X^2 \text{ for all } i \in \Omega\sim\{r,s\}$$
$$\text{and } T_P = \bigcap_{i\in P} T_{\{i\}} \quad \text{for } P \subseteq \Omega \text{ with } |P| \geq 2.$$

Note that $T_P \cap T_Q = T_{P\cup Q}$ for all $P,Q \subseteq \Omega$.

Let $S$ denote the information system constructed in Definition 9 from the atom structure above and let $\mathfrak{B} = \mathfrak{B}_S$ denote the corresponding $KA_\Omega$. Note that $\mathfrak{B}$ is finite when $\mathfrak{A}$ is finite and that $<At\mathfrak{B},\bar{R}^{\mathfrak{B}},\bar{S}^{\mathfrak{B}}> \cong \mathfrak{A}$. Thus, property ($M_0$) holds for the translation given. The finite inseparability of the theory follows.

It follows from 16(iii) and 15.9 of Monk [5] that

COROLLARY 17. For finite $|\Omega| \geq 2$,
(i) the theory of $KA_\Omega$ is undecidable, and
(ii) the theory of all finite $KA_\Omega$'s is undecidable.

Theorem 16(iii) and Corollary 17 can be strengthened by replacing the class $KA_\Omega$ by the class of reduced $KA_\Omega$'s. The stronger result is obtained by interpreting the theory of two disjoint equivalence relations into the class of reduced $KA_\Omega$'s using the same translation.

## References

[1] Comer, S.D., *Finite inseparability of some theories of cylindrification algebras.* **J. Symbolic Logic 34**(1969), 171–176.

[2] Henkin, L., J. D. Monk, and A. Tarski, **Cylindric Algebras, Part I**, North–Holland Pub. Co., Amsterdam, 1971.

[3] Jónsson, B. and Tarski, A., *Boolean algebras with operators. Part I.* **Amer. J. Math. 73**(1951), 891–939.

[4] Lee, T.T., *An algebraic theory of relational databases.* **The Bell System Technical Journal 62**, no. 10, part 2 (December 1983), 3159–3204.

[5] Monk, J. D., **Mathematical Logic**. Springer–Verlag, New York, 1976.

[6] Pawlak, Z., *Information system theoretical foundations,* **Inform. Systems, 6**(1981), 205–218.

[7] Pawlak, Z., *Rough sets and decision tables*, Computation Theory, Lecture Notes in Computer Science, no. 208, edited by A. Skowron, Springer–Verlag, New York, 1985, 187–196.

[8] Pawlak, Z., *Knowledge, reasoning and classification: a rough set perspective.* **EATCS Bulletin**, No. **38**(June 1989), 199–210.

[9] Pudák, P. and J. Tůma, *Every finite lattice can be embedded in the lattice of all equivalences over a finite set,* **Commentationes Math. Univ. Carolinae 18**(1977), 409–414.

[10] Ras, Z. W. and Zemankova, M., *Learning concept descriptions in a growing language.* **Fundamenta Informaticae XII**(1989), 79–96.