Contemporary Mathematics
Volume **131**, 1992 (Part 3)

LATTICES OF CONJUGACY RELATIONS

Stephen D. Comer[1]
The Citadel, Charleston SC 29409, U.S.A.

This paper describes a few elementary properties of the lattice of conjugacy relations of a group. A decomposition of a group into double cosets as well as its decomposition into ordinary conjugacy classes give examples of conjugacy relations. The notion was first defined in the case of groups in Marty [8] to provide examples of hypergroups. An equivalence relation always gives rise to a "quotient" structure. In the case of a conjugacy relation this "quotient" will not necessarily be a group, but a system that we call a polygroup. A polygroup is a special kind of hypergroup in the sense of Marty [8] or multigroup in the sense of Dresher and Ore [7]. Because an isomorphism theorem (THEOREM 4.7) allows us to relate an interval $[\theta,\mathbb{1}]$ in the conjugacy lattice of a group G with the conjugacy lattice of the polygroup "quotient" $G//\theta$, properties for conjugacy relations are developed in the context of a polygroup.

In this paper we will mainly consider conjugacies derivable from subsystems of polygroups and techniques for creating other conjugacies from these. A lot of information about a group is coded into its conjugacy lattice. It is conjuctured that the conjugacy lattice Conj(G) of a finite group G determines the group. To obtain some evidence for this conjecture it is shown that

[1]Research supported in part by a grant from The Citadel Development Foundation. This paper is in final form and no version of it will be submitted for publication elsewhere.
1991 *MR Subject Classification.* Primary 06B99, 20N20; Secondary 06A23

for a finite abelian group G, Conj(G) determines the subgroup lattice of G.

1. Polygroups

In this paper the same symbol is used to denote a function and its obvious extension to sets, eg., we use the symbol f to denote both a function $f: M^k \longrightarrow Sb(M)$ and its natural extension $Sb(M)^k \longrightarrow Sb(M)$ defined by

$$f(X_0,\dots,X_{k-1}) = \cup\{f(x_0,\dots x_{k-1}) : x_i \in X_i \text{ for all } i<k\}$$

for $X_0,\dots,X_{k-1} \subseteq M$.

A *polygroup* is a system $<M,\cdot,^{-1},e>$ where $e \in M$, $^{-1}$ is a unary operation on M, $\cdot$ assigns a nonempty subset of M to each element of $M \times M$, and the following axioms hold for all $x,y,z \in M$:

(P_1) $e \cdot x = \{x\} = x \cdot e$,

(P_2) $x \in y \cdot z$ implies $y \in x \cdot z^{-1}$ and $z \in y^{-1} \cdot x$,

(P_3) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

Note that in (P_3) the extension of $\cdot$ to subsets of M is used. To make reading easier we also identify a singleton subset with its unique element. The product $x \cdot y$ will frequently be denoted by juxtaposition xy.

A polygroup, as defined above, is a system of type <2,1,0>. As with groups, the notion could be defined as a system of type <2,0>. A polygroup is a special case of the notion of hypergroup introduced by Marty [8] and of multigroup due to Dresher and Ore [7].

For a subgroup H of a group G the collection G//H of all double cosets of H forms a polygroup with the natural operations (cf., [7]). A polygroup made from all conjugacy classes of a group G was defined in [8] and discussed in Campaigne [1] and Dietzman

[6]. A dual equivalence between the category of polygroups and the category of complete atomic integral relation algebras is established in Comer [2] where other examples of polygroups are given.

The two examples of polygroups given above are formed from equivalence classes on a group. The first collected double cosets together into an algebraic system and the second collected conjugacy classes together. The notion of a general conjugacy relation, given below, abstracts properties of these two equivalence relations. The notion was first defined, in the case of groups, by Marty [8] who called the relations *conjugations*. This terminology was used by in [1] and [2].

DEFINITION 1.1. *Suppose* $<M,\cdot,^{-1},e>$ *is a polygroup. Then*
(i) *A* ***conjugacy*** *relation on* M *is an equivalence relation on* M *such that for all* $x,y,z,z'\in M$:

(1) $z'\theta z\in x\cdot y$ *implies* $\exists x',y'(\,x'\theta x,\ y'\theta y,$ *and* $z'\in x'\cdot y')$

(2) $x\theta y$ *implies* $x^{-1}\theta y^{-1}$.

(ii) *A conjugacy* θ *on* M *is a* ***special conjugacy*** *if, for all* $x\in M$, $x\theta e$ *implies* $x=e$.

Using the notation $\theta x = \{y\colon y\theta x\}$, a conjugacy relation on M can be described, alternatively, as an equivalence relation θ such that for all $x,y\in M$:

(1′) $\theta(xy) \subseteq (\theta x)(\theta y)$ and

(2′) $(\theta x)^{-1} = \theta(x^{-1})$.

A conjugacy is special if $\theta e = \{e\}$.

We say that N is a *subpolygroup* of a polygroup $<M,\cdot,^{-1},e>$ if $N \subseteq M$, $e\in N$, and for all $x,y\in N$ $x^{-1}\in N$ and $x\cdot y \subseteq N$.

EXAMPLE 1.2. (i) For a subpolygroup H of M define a relation θ_H for $x,y \in M$ by

$$x\theta_H y \text{ iff } HxH = HyH.$$

(ii) If H is a subgroup of Aut(M), define a relation θ^H for x,y∈M by

$$x\theta^H y \text{ iff } \sigma(x) = y \text{ for some } \sigma \in H.$$

The relation θ_H is a conjugacy and the relation θ^H is a special conjugacy on M.

A quotient polygroup can be associated with a conjugacy relation θ on a polygroup M. On the set $\theta M = \{ \theta x : x \in M \}$ of all θ–blocks of an equivalence relation θ on M operations are defined, for x,y∈M, by

(3) $(\theta x)*(\theta y) = \{\theta z : \theta z \subseteq (\theta x)(\theta y) \}$ and

(4) $(\theta x)^{-1} = \theta(x^{-1})$.

The system $<\theta M, *, ^{-1}, \theta e>$, denoted by $M//\theta$, is a polygroup whenever θ is a conjugacy relation. In fact, PROPOSITION 2.1 of [2] shows that the system $M//\theta$, obtained from M using the "induced" operations in (3) and (4), is a polygroup if and only if θ is a conjugacy relation.

A reader should be aware that there are two common ways for an operation · to induce an operation on θM (and make a quotient). One is given in (3) and the other is defined by

(5) $(\theta x)\circ(\theta y) = \{ \theta z : \theta z \cap (\theta x)(\theta z) \neq \emptyset \}$

for all x,y∈M.

The operation defined in (5) is the one normally used to define a quotient structure M/θ of a multivalued algebra, cf., Schweigert [9] or Corsini [5], THEOREM 8. It leads to the notion of a congruence relation on a multivalued algebra. In 3.2 below it is shown that a congruence relation on a polygroup is a conjugacy relation associated with a normal subpolygroup. Of course, there are many other conjugacies.

2. The Conjugacy Lattice

For a polygroup M let Conj(M) denote the collection of all conjugacy relations on M and let $\mathrm{Conj}_s(M)$ denote the collection of all special conjugacies. The smallest conjugacy relation is the identity relation, denoted by δ_M, and the largest conjugacy relation is M^2 which is denoted by $\mathbb{1}_M$. Let $\mathbb{1}_M^s$ denote the special conjugacy relation which identifies all elements of M different from the identity e. When the polygroup M is understood δ, $\mathbb{1}$, and $\mathbb{1}^s$ will be written instead of δ_M, $\mathbb{1}_M$, and $\mathbb{1}_M^s$.

PROPOSITION 2.1. *If M is a polygroup, then*
(i) Conj(*M*) *forms a complete lattice whose join is the same as the join in the lattice of all equivalence relations on M,*
(ii) $\mathrm{Conj}_s(M)$ *is the principal ideal in* Conj(*M*) *determined by* $\mathbb{1}^s$.

Proof. (i). It suffices to show, for every nonempty set S of Conj(M), that the join ΣS of S (in the lattice of all equivalence relations on M) is again a conjugacy. Suppose $z'(\Sigma S)z \in x\cdot y$. Then $z\theta_0 z_1\theta_1 z_2 \ldots \theta_{n-1} z_n = z'$ for some $z_1,\ldots,z_{n-1} \in M$ and $\theta_0,\ldots,\theta_{n-1} \in S$. Because $\theta_0 \in \mathrm{Conj}(M)$, $z_1 \in x_1\cdot y_1$ for some $x_1\theta_0 x$ and $y_1\theta_0 y$ by 1.1(i)(1). Repeat for $\theta_1,\ldots,\theta_{n-1}$ to obtain $x_1,\ldots,x_n$ and $y_1,\ldots,y_n$ such that $x\theta_0 x_1\theta_1\ldots\theta_{n-1}x_n$, $y\theta_0 y_1\theta_1\ldots\theta_{n-1}y_n$ and $z_i \in x_i\cdot y_i$ for all $i \leq n$. Hence $x_n(\Sigma S)x$, $y_n(\Sigma S)y$, and $z' \in x_n\cdot y_n$; so condition (1) of 1.1(i) holds for ΣS. The verification of condition (2) in 1.1(i) is routine.

(ii) is obvious since $\theta \in \mathrm{Conj}(M)$ is special if and only if $\theta \leq \mathbb{1}^s$. □

REMARK 2.2. (i) There are situations when it is desirable to regard a conjugacy on M as a partition of M instead of an equivalence relation. Partitions and equivalence relations will be interchanged freely. Partitions will be written in the form {A;B;...} where A, B, ... are the blocks of the partition, eg., $\{\underline{0};\underline{1,2};\underline{3,4,5}\}$ is the partition with blocks $\{0\}$, $\{1,2\}$, and $\{3,4,5\}$. The largest special conjugacy relation 1_M^s denotes the partition $\{\underline{e};\underline{M\backslash\{e\}}\}$.

(ii) The join and meet in Conj(M) (and $\mathrm{Conj}_s(M)$) are denoted by $\vee$ and $\wedge$, respectively. Neither Conj(M) nor $\mathrm{Conj}_s(M)$ is a sublattice of the partition lattice, in general, because the intersection of two conjugacy relations is not necessarily a conjugacy. We give an example involving conjugacies on S_3. (The complete lattice $\mathrm{Conj}(S_3)$ is given in Fig. 1). Let $S_3 = \{0,1,2,3,\alpha,\beta\}$ where the elements denote the identity permutation, (2,3), (1,3), (1,2), (1,2,3), and (1,3,2) respectively. Then $\theta = \{\underline{0};\underline{2};\underline{\alpha,\beta,1,3}\}$ and $\varphi = \{\underline{0};\underline{3};\underline{\alpha,\beta,1,2}\}$ are special conjugacy relations on S_3 (by the comment proceeding 4.4 below), but $\theta\cap\varphi = \{\underline{0};\underline{2};\underline{3};\underline{\alpha,\beta,1}\}$ is not a conjugacy because $1(\theta\cap\varphi)\alpha = 3\cdot 2$ and $1 \notin 3\cdot 2$. The lemma below and the fact that S_3 is generated by $\{2,3\}$ shows that, for θ and φ above, $\theta\wedge\varphi = \delta$ (= the identity conjugacy relation).

LEMMA 2.3. *If G is a group and* $\theta\in\mathrm{Conj}_s(G)$, *then* $H = \{x\in G : |\theta x| = 1\}$ *is a subgroup of G.*

Proof. Clearly $e\in H$ and H is closed under inverses. If $x,y \in H$, then

$$\theta(xy) \subseteq (\theta x)(\theta y) \subseteq \{xy\}$$

so H is closed under products also. □

3. Conjugacy Relations Determined by Subsystems

In 1.2 a conjugacy relation, θ_H, was associated with a subpolygroup H of a polygroup M. It is shown in 3.2 that these conjugacies include all congruence relations. The definition of a congruence relation on a polygroup given below is a special case of the definition used in [9] for general multialgebras.

DEFINITION 3.1 *An equivalence relation θ on a polygroup M is a* ***congruence relation*** *if for all $x,y,u,v\in M$*

(1) $x\theta y$, $u\theta v$ *implies* $(x\cdot u)\theta(y\cdot v)$, *where for* $A,B\subseteq M$, $A\theta B$ *means that for every* $a\in A$ *there is a* $b\in B$ *such that* $a\theta b$ *and vice–versa.*

(2) $x\theta y$ *implies* $x^{-1}\theta y^{-1}$.

The lattice of all congruences on M is denoted by Con(M). A subpolygroup H of M is *normal* if $xH = Hx$ for all $x\in M$ (cf., Dresher and Ore [7]).

The next result will help to decompose a conjugacy relation into a double coset relation θ_H and a special conjugacy. It is a polygroup version of THEOREM 5 in [3]. It also shows that congruence relations correspond to normal subpolygroups.

THEOREM 3.2. *Suppose M is a polygroup.*

(i) *If $\theta\in\mathrm{Conj}(M)$ and $N=\theta e$, then N is a subpolygroup of M and $\theta_N\subseteq\theta$.*

(ii) *For an equivalence relation θ on M, $\theta\in\mathrm{Con}(M)$ if and only if $\theta=\theta_N$ for some normal subpolygroup N of M.*

Proof. (i) It is clear that $e\in\theta e$ and θe is closed under $^{-1}$ by (2) of 1.1(i). Now suppose $z\in x\cdot y$ where $x,y\in\theta e$. Then, by (P_2), $e\theta x\in z\cdot y^{-1}$ so (1) of 1.1(i) implies $e\in z'\cdot y'$ for some $z'\theta z$ and $y'\theta y^{-1}$. But in a polygroup $e\in z'\cdot y'$ gives $z'=(y')^{-1}$ and $(y^{-1})^{-1}=y$ so $z\theta z'=(y')^{-1}\theta(y^{-1})^{-1}=y\theta e$ which shows $z\theta e$. Thus, θe is a subpolygroup of M. To show $\theta_N\subseteq\theta$, suppose $x\theta_N y$, i.e., $NxN = NyN$. Then, $x\in NxN = NyN\subseteq(\theta e)(\theta y)(\theta e) = \theta y$ since θe is the identity of $M//\theta$. Thus, $x\theta y$ which completes the proof.

(ii) It is straightforward to show that $\theta_N \in \mathrm{Con}(M)$ whenever N is a normal subpolygroup of M ; so ($\Leftarrow$) holds. For ($\Rightarrow$), suppose $\theta \in \mathrm{Con}(M)$. To show $\theta \in \mathrm{Conj}(M)$ it suffices to verify (1) of 1.1(i) so we suppose $z'\theta z \in xy$. Then $x \in zy^{-1}\theta z'y^{-1}$ by (P_2) and $\theta \in \mathrm{Con}(M)$. It follows that there exist $x'\theta x$ with $x' \in z'y^{-1}$, i.e., $z' \in x'y$ using (P_2). Hence $\theta \in \mathrm{Conj}(M)$. Now, by 3.2(i) $\theta \supseteq \theta_N$ where $N = \theta e$ is a subpolygroup of M. Suppose $x\theta y$. Then $e \in x \cdot x^{-1}\theta y \cdot x^{-1}$ by 3.1(1), so there exist $z \in N$ with $z \in yx^{-1}$. Thus, $y \in z \cdot x \subseteq Nx$ which gives $x\theta_N y$. Hence, $\theta = \theta_N$. It remains to show that N is normal. If $y \in Nx$, then $x\theta y$ which implies $e \in x^{-1}x\theta x^{-1}y$ by 3.1(1). Thus, for some $z\theta e$, $z \in x^{-1}y$ which gives $y \in xz \subseteq xN$. Therefore $Nx \subseteq xN$. The other inclusion is similar, so it follows that N is normal. □

By 1.2(i) a conjugacy relation θ_N is associated with every subpolygroup N of M, not just the normal ones. The following summarizes a few properties of this embedding.

PROPOSITION 3.3. *For a polygroup M,*

(i) *The map* $N \longmapsto \theta_N$ *embedds the lattice of subpolygroups of M into* Conj(*M*) *as a poset, in fact as a join semilattice.*

(ii) *The image of the map in* (i) *has only* $\theta_{\{e\}} = \delta$ *in common with* $\mathrm{Conj}_s(M)$. *In particular,* $\mathrm{Con}(M) \cap \mathrm{Conj}_s(M) = \{\delta\}$.

(iii) *If G is a group,* Con(*G*) *is a sublattice of* Conj(*G*).

Proof. (i) Suppose H and K are subpolygroups of M and <H,K> is the subpolygroup generated by H and K. It suffices to show that $\theta_H \vee \theta_K = \theta_{<H,K>}$. If H and K are comparable, say $H \subseteq K$, then $<H,K> = K$ and $\theta_H \vee \theta_K = \theta_K = \theta_{<H,K>}$ clearly holds. Assume that H and K are not comparable. Then $(\theta_H \vee \theta_K)e \supset H$ and $(\theta_H \vee \theta_K)e \supset K$ so $(\theta_H \vee \theta_K)e \supseteq <H,K>$. By 3.2(i), $\theta_H \vee \theta_K \geq \theta_{<H,K>}$. Since the other inclusion is clear, equality holds.

(ii) If $\theta_H \in Conj_s(M)$, then $H = \{e\}$ by 3.2 and $\theta_H = \delta$.
(iii) By a standard group theory argument $\theta_H \cap \theta_K = \theta_{H \cap K}$; so $\theta_H \cap \theta_K$ is a conjugacy relation which equals $\theta_H \wedge \theta_K$. □

The embedding in 3.3 is not, in general, a lattice embedding. For example in S_3 (using the notation in 2.2(ii)), if $H = \{0,1\}$ and $K = \{0,\alpha,\beta\}$, then $H \cap K = \{0\}$ but $\theta_H \cap \theta_K = \{\underline{0};\underline{1};\underline{\alpha,\beta};\underline{2,3}\} \in Conj(S_3)$ which gives $\theta_H \wedge \theta_K \neq \delta = \theta_{H \cap K}$.

Certain extreme elements in Conj(M) can be described in terms of conjugacies related to subpolygroups. The dual atoms and the minimal non–special elements in Conj(M) are described below. If N is a subpolygroup of M, the partition $\{\underline{N};\underline{M \setminus N}\}$ is a conjugacy relation on M which we denote by $\bar{\theta}_N$. For an arbitrary $\varphi \in Conj(M)$ let $\bar{\varphi} = \bar{\theta}_N$ where $N = \varphi e$. Observe that $\bar{\varphi} = \mathbb{1}^S$ whenever φ is special.

Proposition 3.4. (i) *The dual atoms in* Conj(*M*) *are exactly the conjugacy relations* $\bar{\theta}_N$ *where N is a proper subpolygroup of M.*

(ii) *The minimal elements of the poset* $\{ \theta \in Conj(M) : \theta \nleq \mathbb{1}^S \}$ *are the elements* θ_N *where N is an atom in the subalgebra lattice of M.*

Proof. (i). Suppose θ is a dual atom of Conj(M). $N = \theta e$ is a proper subpolygroup of M, by 3.2(i) and $\theta \leq \bar{\theta}_N$. Since θ is a dual atom $\theta = \bar{\theta}_N$.
(ii). If θ is not a special conjugacy relation, $N = \theta e \neq \{e\}$ and, by 3.2(i), $\theta_N \leq \theta$. If θ is minimal and not special, then $\theta_N = \theta$. □

Corollary 3.5. (i) *The number of dual atoms in* Conj(*M*) *is equal to the number of proper subpolygroups of M. In particular,* $\mathbb{1}^S = \bar{\theta}_{\{e\}}$.

(ii) *for a group* G, $\mathbb{1}$ *is* $\vee$*-irreducible in* $\mathrm{Conj}(G)$ *iff* G *is a simple abelian group.*

4. Splitting Conjugacy Relations

In this section we describe techniques for investigating intervals in Conj(M) that lie above or below a given non–special conjugacy relation.

By 3.2(i) the block θe of a conjugacy θ is a subpolygroup. A new conjugacy $\theta[\varphi]$ may be obtained from θ by replacing the θe block by the blocks of a conjugacy $\varphi \in \mathrm{Conj}(\theta e)$. More precisely,

DEFINITION 4.1. *For* $\theta \in \mathrm{Conj}(M)$ and $\varphi \in \mathrm{Conj}(\theta e)$, *the* φ*–**split** of* θ *is an equivalence relation* $\theta[\varphi]$ *on* M *defined by*

$$\theta[\varphi]x = \begin{cases} \theta x & \text{if } \theta x \neq \theta e \\ \varphi x & \text{if } \theta x = \theta e \end{cases}.$$

PROPOSITION 4.2. $\theta[\varphi] \in \mathrm{Conj}(M)$. *Moreover,* $M//(\theta[\varphi])$ *is isomorphic to* $(\theta e//\varphi)[M//\theta]$, *the polygroup extension of* $\theta e//\varphi$ *by* $M//\theta$ *introduced in* [4].

Proof. To verify the first statement it suffices to show that the product of two $\theta[\varphi]$–blocks is a union of $\theta[\varphi]$–blocks. Along the way we develop a rule for computing the product of two $\theta[\varphi]$–blocks from which the isomorphism is apparent. Let $\theta[\varphi] = \psi$ for short. The first two cases are obvious from the definition of $\theta[\varphi]$:

(1) $(\psi x)(\psi y) = (\varphi x)(\varphi y)$ if $\theta x = \theta y = \theta e$

(2) $(\psi x)(\psi y) = (\theta x)(\theta y)$ if $\theta x, \theta y \neq \theta e$.

When computing these products, replace θe by $\{ \varphi x : x \in \theta e \}$. For the other cases,

(3) $(\psi x)(\psi y) = \theta y$ if $\theta x = \theta e \neq \theta y$

(4) $(\psi x)(\psi y) = \theta x$ if $\theta y = \theta e \neq \theta x$.

To verify (3) first note that $(\varphi x)(\theta y) \subseteq (\theta e)(\theta y) = \theta y$. Now, suppose $y' \theta y$. Then $y' \in x \cdot z$ for some z (using (P_2)); so $y' \in x(\theta z) \subseteq (\theta e)(\theta z) = (\theta z)$. Then $(\theta y) \cap (\theta z) \neq \emptyset$ so $\theta z = \theta y$ which gives $\theta y \subseteq x(\theta y) \subseteq (\varphi x)(\theta y)$ as desired. The proof of (4) is similar so $\theta[\varphi]$ is a conjugacy relation. The isomorphism is established by comparing (1), (2), (3), (4) with the definition of the product on the polygroup $(\theta e//\varphi)[M//\theta]$. □

For $\varphi \leq \psi$ in Conj(M) let $[\varphi,\psi]$ denote the interval $\{ \theta : \varphi \leq \theta \leq \psi \}$ in Conj(M). The map $\varphi \longmapsto \theta[\varphi]$ immediately gives

COROLLARY 4.3. Conj(θe) *is isomorphic to the interval* $[\theta[\delta_{\theta e}],\theta]$ *in* Conj(M).

For a subset X of a polygroup such that $e \in X$ we let $X^* = X \backslash \{e\}$. There is one splitting of a conjugacy θ that deserves special attention. Namely, for a conjugacy θ let

$$\theta^S = \theta[1^S_{\theta e}]$$

where $1^S_{\theta e}$ is the unity element in $\mathrm{Conj}_S(\theta e)$. In other words, θ^S is a special conjugacy (by 4.2) obtained from θ by splitting θe into the two classes: $\{e\}$ and $(\theta e)^*$.

The splitting operation above is a useful way to show that certain equivalence relations are conjugacies. For example, consider the equivalence relation $\theta = \{\underline{0};\underline{2};\underline{\alpha,\beta,1,3}\}$ on S_3 used in 2.2(ii). It follows that $\theta \in \mathrm{Conj}(S_3)$ because $\theta = \bar{\theta}^S_H$ where $H = \{0,2\}$ is a subgroup of S_3.

A few elementary properties of θ^S are given below.

PROPOSITION 4.4. *For* $\theta,\varphi \in \mathrm{Conj}(M)$

(i) $\theta^S = \theta$ *if* θ *is special and* $\theta^S = \theta \cap 1^S$ *if* θ *is not special,*

(ii) $\theta^S \leq \theta$,

(iii) θ *covers* θ^S in $\mathrm{Conj}(M)$ *if* θ *is not special,*

(iv) $\varphi \leq \theta$ *implies* $\varphi^S \leq \theta^S$,

(v) θ *is determined by* θ^S *and* θe. *Namely,* $\theta = \theta_N | \theta^S$, *a commuting join, where* $N = \theta e$.

Proof. (v). Suppose $x\theta y$. If $\theta x \neq \theta e$, $\theta^S x = \theta x$ so $x\theta_N x \theta^S y$ and if $\theta x = \theta e$ (=N), $x\theta_N y \theta^S y$. Thus, $\theta \leq \theta_N | \theta^S$. □

Information about the structure of Conj(M) can be obtained from 4.4(v). For example, if G is an Abelian group, every $\theta \in \mathrm{Conj}(G)$ is a join of a congruence relation and a special conjugacy. In particular, if θ is $\vee$–irreducible in Conj(G), either θ is $\vee$–irreducible congruence relation or $\vee$–irreducible special conjugacy. The converse does not hold, for example, $\mathrm{Con}(\mathbb{Z}_8)$ is a 4 element chain but $1 = \theta_{<2>} \vee 1^S$ is not $\vee$–irreducible in $\mathrm{Conj}(\mathbb{Z}_8)$. (See figure 2.)

The map $\theta \longmapsto \theta^S$ is shown to be a lattice homomorphism of Conj(M) onto $\mathrm{Conj}_S(M)$ in 4.6 below. Towards this goal the following lemma about lattices is needed.

LEMMA 4.5. *If h is a join retract of a lattice L onto an ideal of L (ie.,* $h\colon L \longrightarrow L$ *satisfies* $h(x \vee y) = (hx) \vee (hy)$, $hx \leq x$, $h(hx) = hx$ *for all* $x,y \in L$ *and* $h(L)$ *is an ideal of* L*), then h is a homomorphism.*

Proof. Since h preserves order, $h(x \wedge y) \leq (hx) \wedge (hy)$. If $z \leq (hx) \wedge (hy)$, then $z \leq hx \leq x$ and $z \leq hy \leq y$ so $z \leq x \wedge y$. Hence $z = hz \leq h(x \wedge y)$ because $z \leq hx \in h(L)$ implies $z \in h(L)$ and h fixes elements of h(L). Thus, $(hx) \wedge (hy) = h(x \wedge y)$. □

PROPOSITION 4.6. *The map* $\theta \longmapsto \theta^s$ *is a lattice homomorphism of* $\mathrm{Conj}(M)$ *onto* $\mathrm{Conj}_s(M)$.

Proof. Applying 4.5, by 4.4(i),(ii) and 2.1(ii), it suffices to show the map preserves joins. Since $\leq$ is preserved by 4.4(iv) we only need to show that $(\theta\vee\varphi)^s \leq \theta^s\vee\varphi^s$ in Conj(M). Suppose $x(\theta\vee\varphi)^s y$ and $x,y \neq e$. Then there exist a sequence $x = x_0,\ldots,x_n = y$ such that $x_0\theta x_1\varphi x_2\ldots x_{i-1}\theta x_i\varphi x_{i+1}\ldots x_n$. If $x_j\theta x_{j+1}$ and $x_j, x_{j+1} \neq e$, then $x_j\theta^s x_{j+1}$ and similarly for φ. Hence we may assume $x_i = e$ and $x_{i-1}, x_{i+1} \neq e$ for some i. Then $x_{i-1} \in H = \theta e$ and $x_{i+1} \in K = \varphi e$. If $x_{i+1} \in H$, then $x_{i-1}\theta x_{i+1}\theta x_{i+2}$ and we may drop $e = x_i$ from the sequence. Hence we may assume $x_{i+1} \notin H$. Then $e \notin x_{i-1}\cdot x_{i+1}$ because if so, $x_{i+1} = x_{i-1}^{-1} \in H$. Choose $x_i' \in x_{i-1}\cdot x_{i+1}$. Then $x_i' \in Hx_{i+1} \subseteq Hx_{i+1}H \subseteq \theta x_{i+1}$ and $x_i' \in x_{i-1}K \subseteq Kx_{i-1}K \subseteq \varphi x_{i-1}$ so $x_{i-2}\varphi x_{i-1}\varphi x_i'\theta x_{i+1}\theta x_{i+2}$ which means we can shorten the sequence from x_0 to x_n and eliminate the term $x_i = e$. Repeating the above for all $x_i = e$ we obtain $x = x_0'\theta x_1'\varphi\ldots x_m' = y$ where all $x_i' \neq e$. Thus, $(x,y) \in \theta^s\vee\varphi^s$. □

For $\varphi,\theta\in\mathrm{Conj}(M)$ and $\varphi \subseteq \theta$, we define a conjugacy $\theta//\varphi$ on $M//\varphi$ by

$$(\varphi x)(\theta//\varphi)(\varphi y) \iff x\theta y$$

for all $x,y\in M$. The first part of the following theorem gives a lattice version of the First Isomorphism Theorem from group theory.

THEOREM 4.7. *Suppose* $\varphi \in \mathrm{Conj}(M)$. *Then*

(i) *The map* $\theta \longmapsto \theta//\varphi$ *is an isomorphism of the interval* $[\varphi,\mathbb{1}]$ *in* $\mathrm{Conj}(M)$ *onto* $\mathrm{Conj}(M//\varphi)$.

(ii) $\theta//\varphi$ *is special in* $\mathrm{Conj}(M//\varphi)$ *iff* $\theta e = \varphi e$. *Moreover, the map in* (i) *is an isomorphism of the interval* $[\varphi,\overline{\varphi}]$ *in* $\mathrm{Conj}(M)$ *onto* $\mathrm{Conj}_s(M//\varphi)$.

(iii) *The map* $\theta \longmapsto \theta^s$ *is an isomorphism of* $[\varphi,\overline{\varphi}]$ *in* Conj(M) *onto* $[\varphi^s,\overline{\varphi}^s]$ *in* Conj$_s(M)$.

(iv) *If* φ *is not special, the map* $\theta \longmapsto \theta^s$ *is an isomorphism* $[\varphi,\mathbb{1}] \cong [\varphi^s,\mathbb{1}^s]$.

(v) *If* N *is a subpolygroup of* M, Conj$_s(N) \cong [\theta_N[\delta],\theta_N^s]$, *a sublattice of* Conj$_s(M)$.

Proof. (i) is a tedious but straightforward argument.

(ii). $\theta//\varphi$ is special iff $(\varphi x)(\theta//\varphi)(\varphi e) \implies \varphi x = \varphi e$ iff $x\theta e t \implies x\varphi e$ iff $\theta e \subseteq \varphi e$. But $\varphi e \subseteq \theta e$ always holds since $\varphi \subseteq \theta$. Since $\theta \in [\varphi,\overline{\varphi}]$ iff $\varphi e \subseteq \theta e$, the restriction of the map in (i) gives the desired isomorphism.

(iii). If φ is special, $\theta^s = \theta$ for θ in $[\varphi,\overline{\varphi}] \subseteq$ Conj$_s$(M). Assume $\varphi e \neq \{e\}$. Since $\theta \longmapsto \theta^s$ is a lattice homomorphism it suffices to show the map is (1) one–one and (2) onto. For (1) suppose $\theta_1 \neq \theta_2$ in $[\varphi,\overline{\varphi}]$. Since $\theta_1 e = \varphi e = \theta_2 e$, there exist $x \notin \varphi e$ such that $\theta_1 x \neq \theta_2 x$. By 4.4(i), $\theta_1^s x \neq \theta_2^s x$ so the images are distinct and thus (1) holds. For (2) assume θ is in $[\varphi^s,\overline{\varphi}^s]$. Define $\theta^+ = \{\underline{\varphi e};\underline{\theta x_1};\ldots\}$ where $\theta = \{\underline{e};\underline{(\varphi e)^*};\underline{\theta x_1};\ldots\}$. Since $(\theta^+)^s = \theta$ it suffices to show that θ^+ is a conjugacy. If $x\theta^+ y$, it is clear that $x^{-1}\theta^+ y^{-1}$ since $(\varphi e)^{-1} = \varphi e$ and $(\theta x_i)^{-1} = \theta(x_i^{-1})$ for all i; thus, condition (2) of 1.1(i) holds. To verify (1) in 1.1(i) we need to show $(\theta^+ x)(\theta^+ y)$ is a union of θ^+–blocks. First we show $(\varphi e)(\theta x_i) = \theta x_i$. This holds because $\theta \supseteq \varphi^s$ and $\theta x_i \neq \varphi e$ implies that θx_i is a union of φ–blocks $\theta x_i = (\varphi x_i) \cup \ldots \cup (\varphi x_i') \cup \ldots$ and $(\varphi e)(\varphi x_i') = \varphi x_i'$ for each component $\varphi x_i'$. It remains to see that $(\theta x_i)(\theta x_j)$ is a union of θ^+–blocks. For this it suffices to show

$$(\varphi e)^* \subseteq (\theta x_i)(\theta x_j) \quad \text{iff} \quad e \in (\theta x_i)(\theta x_j).$$

For ($\Longrightarrow$) choose $x \in (\varphi e)^*$. Then $x \in x_i' \cdot x_j'$ for some $x_i' \theta x_i$, $x_j' \theta x_j$. Since $\theta x = \varphi^s x \subseteq (\varphi^s x_i')(\varphi^s x_j') = (\varphi x_i')(\varphi x_j')$, $e\varphi x$, and φ is a conjugacy, $e \in (\varphi x_i')(\varphi x_j') \subseteq (\theta x_i)(\theta x_j)$. The implication ($\Longleftarrow$) is similar, so $\theta^+ \in \mathrm{Conj}(M)$ which completes the proof of (iii).
(iv) holds by an argument similar to (iii).
(v) follows from 4.3 and the observation that $\theta_N[\varphi]$ is special iff φ is special. □

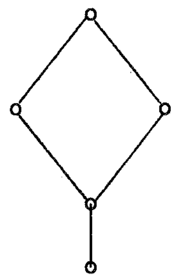
In 4.7(iv) if $\varphi \in \mathrm{Conj}_s(M)$, the homomorphism is, in general, not one–one.

5. Epilogue

In sections 3 and 4 (e.g., 3.4, 3.5, and 4.2) several techniques were given for creating conjugacy relations on a polygroup. Also, several isomorphisms (e.g., 4.3, 4.6, 4.7) were established which allow for the study of parts of Conj(M). The techniques described in the previous sections play a fundamental role in the proofs of the following two results. The proofs are extremely long and are omitted.

THEOREM 5.1. *For a group G,* Conj(G) *is a modular lattice if and only if* $G \cong \mathbb{Z}_n$ *for* $n = 1,2,3,4,5,7$. *Moreover,* $\mathrm{Conj}_s(G)$ *is a modular lattice but* Conj(G) *is not modular if and only if* $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

THEOREM 5.2. *There exist a lattice formula that defines* $\mathbb{1}^s$ *in* Conj(G) *for all groups* G *not isomorphic to* $\mathbb{Z}_4$. *On the other hand,* $G \cong \mathbb{Z}_4$ *if and only if* Conj(G) *has the form*



We conclude with an application of 5.2 which uses several of the results developed in sections 3 and 4.

THEOREM 5.3. *If* G *and* G' *are finite Abelian groups and* $\mathrm{Conj}(G) \cong \mathrm{Conj}(G')$, *then* $\mathrm{Con}(G) \cong \mathrm{Con}(G')$, *i.e., the subgroup lattices of* G *and* G' *are isomorphic.*

Proof. The proof breaks into two cases: (1) G is not isomorphic to $\mathbb{Z}_{2^k}$ for any k, and (2) $G \cong \mathbb{Z}_{2^k}$ for some k.
Assume (1) and let $f:\mathrm{Conj}(G) \longrightarrow \mathrm{Conj}(G')$ be an isomorphism onto $\mathrm{Conj}(G')$. We want to show that the restriction of f to the sublattice Con(G) is an isomorphism onto Con(G′). By induction on the height of θ_H in Con(G), we prove that $f(\theta_H) \in \mathrm{Con}(G')$ and that f is an isomorphism of $[\theta_H, \mathbb{1}]$ onto $[f\theta_H, \mathbb{1}]$. This is clear for $\theta_{\{e\}}$ which is the smallest element of both lattices. If G is simple, $\mathbb{1}$ is $\vee$–irreducible and $\mathrm{Con}(G) = \{\delta, \mathbb{1}\}$ by 3.5(ii). Therefore G′ is simple and clearly $f\mathbb{1} = \mathbb{1}$. Assume G is not simple. Since G is not isomorphic to $\mathbb{Z}_4$, $\mathbb{1}^s$ is definable in Conj(G) and also in Conj(G′) by 5.2. Thus $f(\mathbb{1}^s_G) = \mathbb{1}^s_{G'}$. Therefore, by 3.4(ii) f carries each atom θ_H in Con(G) to an atom $f(\theta_H)$ in Con(G′). Thus the conclusion is true for all θ_H of height 1 in Con(G). Suppose the result is true for all θ_H of height $\leq n$ and consider elements of height n+1. Since such an element covers a θ_H of height n, we may consider θ_K as an atom in the lattice $[\theta_H, \mathbb{1}]$ which is isomorphic by f to $[f\theta_H, \mathbb{1}] \subseteq$

Conj(G′). If G/H is not isomorphic to $\mathbb{Z}_4$, then $\mathbb{1}^s$ is definable in $[\theta_H,\mathbb{1}] \cong$ Conj(G/H), by 4.7(i), and also in $[f\theta_H,\mathbb{1}]$. In this case, by 3.4(ii) θ_K is mapped to a corresponding element in Con(G′). Now, suppose $G/H \cong \mathbb{Z}_4$. The nontrivial proper subgroup of $\mathbb{Z}_4$ corresponds to a unique element θ_K in Con(G) that covers θ_H. We claim there exist a subgroup L of G incomparable to K. (For if not, every subgroup of G is comparable with K which implies $G \cong \mathbb{Z}_{2^k}$ for some k by the Fundamental Theorem of Finite Abelian Groups.) Now, $L \not\subseteq H$ because $L \not\subseteq K$; so $H \vee L \geq K$ since K is the only cover of H in the subgroup lattice of G. By the modular law $H \vee (L \cap K) = K \cap (H \vee L) = K$ because $K \geq H$. Also, $H \not\subseteq L$ because only K and G extend H; so $K \cap L \neq H$ and $\theta_{K\cap L}$ has height $\leq n$. Since f is a lattice isomorphism, $f(\theta_K) = f(\theta_H) \vee f(\theta_{L\cap K})$ belongs to Con(G′) because $f(\theta_H)$ and $f(\theta_{L\cap K})$ belong by the induction assumption. Moreover, since $G/H \cong \mathbb{Z}_4$, f restricted to $[\theta_K,\mathbb{1}]$ is an isomorphism. This completes the proof of (1).

(2) is proved by a straight forward induction on k. □

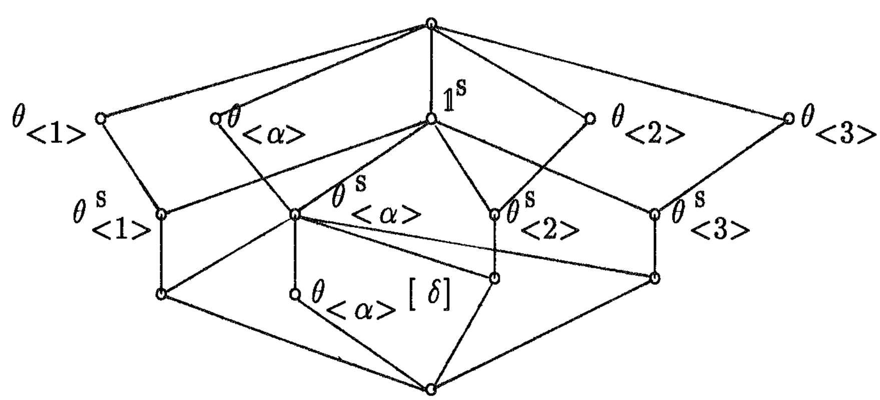


Figure 1, Conj(S_3)

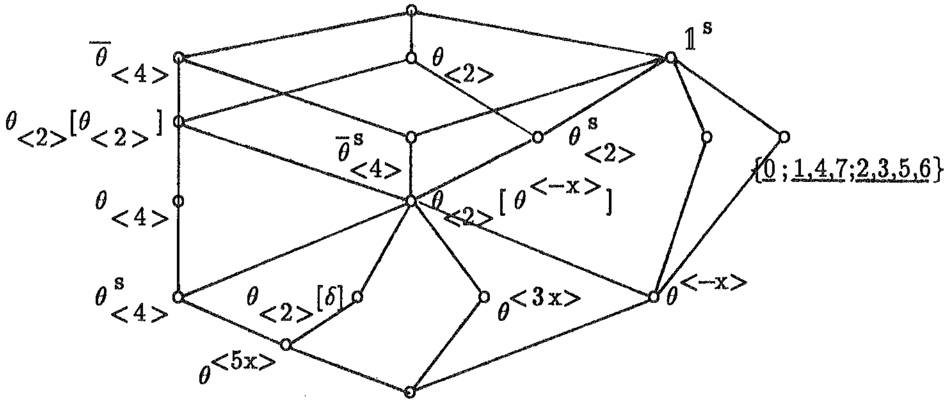


Figure 2, Conj($\mathbb{Z}_8$)

Bibliography

1. Campaigne, H., *Partition Hypergroups.* **Amer. J. Math.** 62(1940), 599–612.
2. Comer, S., *Combinatorial Aspects of Relations.* **Algebra Univ.** 18(1984), 77–94.
3. Comer, S., *Polygroups Derived from Cogroups.* **J. Algebra** 89(1984), 397–405.
4. Comer, S., *Extensions of polygroups by polygroups and their representations using color schemes.* In, **Universal Algebra and Lattice Theory** (R.S. Freese and O.C. Garcia, editors), Lect. Notes in Math. #1004, Springer–Verlag, 1983, 91–103.
5. Corsini, P. *Prolegomeni alla Theoria degli ipergruppi.* Quaderni dell' institudo di Math, Inform. e Sist (Udine), 1986, 127 pp.
6. Dietzman, P. *On the multigroups of complete conjugate sets of elements of a group.* **C.R. (Doklady) Acad. Sci. URSS (N.S.)** 49(1946), 315–317.
7. Dresher, M. and O. Ore, *Theory of Multigroups.* **Amer. J. Math.** 60(1938), 705–733.
8. Marty, F., *Sur une generalization de la notion de groupe.* **Huitieme Congrès des mathématiciens scand.**, Stockholm, 1934, 45–49.
9. Schweigert, D., *Congruence Relations of Multialgebras.* **Discrete Math.** 53(1985), 249–253.