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APPLICATION-ORIENTED HYPERSTRUCTURES[1]

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Abstract: This paper describes several hypergroup constructions based on considerations which arise in the study of symmetry. In particular, we show that a natural hypergroup is associated with every character algebra.

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1. Introduction.

The study of symmetry is closely related to the theory of groups and its extensions. Symmetry groups have been widely applied in chemistry [6], crystallography [8], and solid-state physics [1]. These applications have involved coset decompositions, double coset decompositions, decompositions into conjugacy classes, and group characters. Below we introduce a hypergroup structure on each of these "spaces". In general, we show that a C-algebra in the sense of Y. Kawada [9] gives raise to a special hypergroup called a quasi-cannonical hypergroup by Bonansinga and Corsini [4] and a polygroup in [5].

A *polygroup* is a system $\langle M, \bullet, e\rangle$ where $e \in M$, $\bullet$ assigns a nonempty subset of M to each pair of elements in M, and the following axioms hold for all $x, y, z \in M$:

(1) for each x there exist a unique $x^{-1} \in M$ such that $e \in x \bullet x^{-1}$ and $e \in x^{-1} \bullet x$,

(2) $e \bullet x = \{x\} = x \bullet e$,

(3) $x \in y \bullet z$ implies $y \in x \bullet z^{-1}$ and $z \in y^{-1} \bullet x$, and

(4) $(x \bullet y) \bullet z = x \bullet (y \bullet z)$.

2. Classical hypergroup constructions.

Coset Hypergroups. For a group G and a subgroup H let G/H denote the collection of all right cosets $Hg = \{hg: h \in H\}$ for g in G. The system $\langle G/H, \bullet, H\rangle$ is a hypergroup where

$$(Hg_1) \bullet (Hg_2) = \{Hg : g \in g_1 H g_2\}.$$

The system G/H is called a *D-hypergroup* and is the principal example of a cogroup. Recently Y. Sureau [13] has characterized all cogroups by nested triples of permutation groups.

The relationship of cosets to the study of color symmetry has its origin in the work of Shubnikov [12]. See also, Roth [11]. The idea is to select a *fundamental region* Ω of a design (or crystal); then, for each element g in the symmetry group G of the design, label the region Ωg with the element g. (Of course, not all regions of the design may be labeled.) A coloring of the design is induced by a subgroup H of G by assigning colors to the right cosets of H and coloring the regions labeled by the members of the coset with the color assigned to the coset. The color scheme is consistent in the sense that symmetries map regions with a given color onto commonly colored regions. The cogroup G/H, with colors replacing right cosets, gives information about the coloring scheme.

Double coset algebras. The unit element of a cogroup is normally a one sided scalar and lacks symmetry. This is overcome by looking at double cosets. A double coset HgH of a subgroup H of a group G is the collection of all distinct elements h_1gh_2 where h_1 and h_2 range over H. The double cosets of H give a decomposition of G. Let $G//H$ denote the collection of all double cosets of G with respect to H. A double coset algebra is a hypergroup $\langle G//H, \bullet, H\rangle$ where

$$(Hg_1H)\bullet(Hg_2H)=\{Hg_1hg_2H : h\in H\}.$$

The system G//H is a example of a polygroup, i.e., quasi-cannonical hypergroup.

Conjugacy class polygroups. In dealing with a symmetry group two symmetric operations belong to the same class if they represent the same map with respect to (possibly) different coordinate systems where one coordinate system is converted into the other by a member of the group (cf., [6]). In the language of group theory this means elements a, b in a symmetry group G belong to the same class if there exist a $g\in$ G such that $a = gbg^{-1}$, i.e., a and b are conjugate. The collection of all conjugacy classes of a group G is denoted by $\bar{G}$ and the system $\langle\bar{G},\bullet,\{e\}\rangle$ is a polygroup where e is the identity of G and the product $A\bullet B$ of conjugacy classes A and B consists of all conjugacy classes contained in the elementwise product AB.

Character polygroups. Closely related to the conjugacy classes of a group are its characters. Let $\hat{G} = \{\chi_1, \chi_2, \cdots, \chi_k\}$ be the collection of irreducible characters of a finite group G where χ_1 is the trivial character. The character polygroup $\hat{G}$ of G is the system $\langle\hat{G}, \bullet, \chi_1\rangle$ where the product $\chi_i\bullet\chi_j$ is the set of irreducible components in the elementwise product $\chi_i\chi_j$. The system $\hat{G}$ was investigated by R. Roth [10] who considered a duality between $\hat{G}$ and $\bar{G}$.

3. A construction from C-algebras.

The above constructions of hypergroups based on double cosets, conjugacy classes, and characters all make use of the fact that when two objects of a type under consideration (i.e., double cosets, characters, etc) are multiplied in the natural way, the product is uniquely composed of objects in the set (perhaps with repetition). The polygroup binary operation is a type of convolution which tells which objects are in the product but not the multiplicity. We formalize this idea by showing that every C-algebra gives raise to a polygroup. The notion of C-algebra (or character algebra) presented here is due to Y. Kawada [9] (see also

[2]) except that the commutativity requirement has been weakened.

A *C-algebra* is a classical algebra A together with a basis $X = \{x_0, \cdots, x_d\}$ for A (as a complex linear space) such that

(C$_1$) A is an algebra and $x_i \bullet x_j = \sum_k p_{ij}^k x_k$ for all i, j,

(C$_2$) A has an identity element $e = x_0$, i.e., $p_{0j}^k = \delta_{jk} = p_{j0}^k$,

(C$_3$) every p_{ij}^k is a real number,

(C$_4$) there exist a permutation $i \to i'$ (for $i = 0, ..., d$) such that $(i')' = i$ and $p_{ij}^k = p_{j'i'}^{k'}$,

(C$_5$) $p_{ji}^0 = p_{ij}^0 = \kappa_i \delta_{ij'}$ with $\kappa_i > 0$ for all i, j, and

(C$_6$) the map $x_i \mapsto \kappa_i$ induces a linear representation of A.

The condition (C$_4$) implies that the map $x_i \mapsto x_{i'}$ extends to an anti-automorphism of A. A C-algebra is *commutative* if $p_{ij}^k = p_{ji}^k$ for all i, j, k. The lemma below summarizes a few elementary facts.

Lemma. (1) $0' = 0$,

(2) $\kappa_0 = 1$,

(3) $\kappa_i = \kappa_{i'}$,

(4) $\kappa_k p_{ij}^k = \kappa_i p_{kj'}^i = \kappa_j p_{i'k}^j$.

Proof. (1) and (2) both follow from (C$_2$) and (C$_5$). (C$_5$) implies $\kappa_i \delta_{ij'} = \kappa_j \delta_{ji'}$ for all i, j; so $\kappa_i = \kappa_j$ when $j = i'$ and thus (3) holds. To see the first equality in (4) express both sides of $(x_i \bullet x_j) \bullet x_{k'} = x_i \bullet (x_j \bullet x_{k'})$ as a linear combination of $x_0, \cdots, x_d$ and compare the coefficients of x_0. The first equality, (C$_4$) and (3) yield the second:

$$\kappa_i p_{kj'}^i = \kappa_{i'} p_{jk'}^{i'} = \kappa_j p_{i'k}^j \quad . \qquad \square$$

Proposition. Every C-algebra A with basis X such that the parameters p_{ij}^k are all non-negative (the Kreim condition) determines a polygroup $Pg(A) = \langle X, \bullet, e \rangle$ where $x_i \bullet x_j = \{x_k : p_{ij}^k \neq 0\}$ and $x_i^{-1} = x_{i'}$ for all i, j.

Proof. Since $x_i \bullet x_{i'} = \sum_k p_{ii'}^k x_k = \kappa_i x_0 + \cdots$, it is clear that $x_0 \in x_i \bullet x_{i'}$. If $x_0 \in$

$x_i \bullet x_{j'}$, $p_{ij}^{0} \neq 0$ which implies $j = i'$ by (C_5). Similarly, $x_{i'}$ is the only y such that $x_0 \in y \bullet x_i$, so axiom (1) holds. Axiom (2) follows from (C_2) and axiom (3) from Lemma (4). For (4) notice that $x_u \in (x_i \bullet x_j) \bullet x_k$ if and only if $p_{ij}^{v} p_{vk}^{u} \neq 0$ for some v and similarly, $x_u \in x_i \bullet (x_j \bullet x_k)$ if and only if $p_{iv}^{u} p_{jk}^{v} \neq 0$ for some v. The associative law for $Pg(A)$ follows from the equality $\sum_v p_{ij}^{v} p_{vk}^{u} = \sum_v p_{iv}^{u} p_{jk}^{v}$ (a consequence of (C_1)) and the Krein condition. □

Examples of C-algebras not only include the situations mentioned earlier, but also the adjacency algebras of association schemes ([2]), S-algebras over finite groups ([3]), and centralizer algebras of homogeneous coherent configurations ([7]). The centralizer algebra of a coherent configuration is called a cellular algebra in the work of Weisfeiler ([14]) on the graph isomorphism problem.

References

[1]. Altmann, S., *Band theory of solids: An introduction from the point of view of symmetry*, Oxford University Press, 1991

[2]. Bannai, E. and T. Ito, *Algebraic Combinatorics I: Association Schemes,* Benjamin Cummings, 1984.

[3]. Brender, M., *A class of Schur algebras,* Trans. AMS **248**(1979), 435-444.

[4]. Bonansinga, P. and P. Corsini, *Sugli omomorfismi di semi-ipergruppi e di ipergruppi*. Boll. Un. Mat. Italy **1-B**(1982), 717-727.

[5]. Comer, S., *Combinatorial aspects of relations*. Algebra Univ. **18**(1984), 77-94.

[6]. Cotton, F. Albert, *Chemical Applications of Group Theory*. J. Wiley and Sons, 1990.

[7]. Higman, D.G., *Combinatorial considerations about permutation groups,* Lecture Notes, Math. Institute, Oxford, 1972.

[8]. Janovec, V., E. Dvorakova, T.R. Wike, and D.B. Litvin, *The coset and double coset decompositions of the 32 crystallographic point groups.* Acta Cryst. Sect. A **45**(1989), no.11, 801-802.

[9]. Kawada, Y., *Über den Dualitätssatz der Charaktere nichtcommutativer Gruppen*. Proc. Phys. Math. Soc. Japan **24**(1942), 97-109.

[10]. Roth, R., *Character and Conjugacy Class Hypergroups of a Finite Group,* Annali di Matematica pura ed applicata. **105**(1975), 295-311.

[11]. Roth, R., *Color Symmetry and Group Theory*, Discrete Math. **38**(1982), 273-296.
[12]. Shubnikov, A.V. and V.A. Koptsik, *Symmetry in Science and Art* (translated from the Russion), Plenum Press, 1974.
[13]. Sureau, Y., *Groupes associés à un cogroupe*, Algebraic Hyperstructures and Applications, Hadronic Press, 1994, 35-44.
[14]. Weisfeiler, B., On construction and identification of graphs, Lecture Notes in Math. #558, Springer-Verlag, 1976.