Plane vs Spherical Geometry

Finding Area

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I. INTRODUCTION

1. How big is South America?
2. How big is Antartica?

The first is approximately a triangle on the globe and the second is approximately a circle.

II. PLANE AREA VS SPHERICAL AREA

A. Measuring area in the plane

The basic unit is a square and square units (in^2, ft^2, km^2, etc) are used. The idea is to cover a region with squares and count them. The standard formulas are:

Area of rectangle = base*height
Area of triangle = (1/2)*base*height
Area circle = π*radius^2

B. Measuring area on the sphere - the problem.

Fundamental difficulty - There are NO rectangles (so not squares) on the sphere!! There are no natural square units to use. While a proof of this is not elementary, it is possible to develop faith in the statement by a few explorations on the sphere. The following is an example.

EXPLORATION 1. Try constructing a spherical "square".
Idea: take a method for constructing a square on the plane and try it on the sphere. For example,

1. In the plane,
   a. Draw a circle
   b. Draw two perpendicular diameters and lable their intersections with the circle as A, B, C, and D.
   c. Draw the "sides" (AB, BC, CD, and DA), measure them, and the angles A, B, C, and D.
   d. Compare the lengths of the sides and the angles.

2. On the sphere, repeat the steps (a)-(d) from above[1]. You should find that the sides are congruent and that the angles are congruent, but not right angles.

C. What is area?

The area of a region is a real number associated with that region - this association is a function A(region) with the following properties.

(1) $A(R) > 0$ for every region R.
(2) The interior of congruent regions have the same area, e.g., if $\Delta ABC \cong \Delta DEF$, then $A(int(\Delta ABC)) = A(int(\Delta DEF))$.
(3) If R_1 and R_2 are disjoint regions, then $A(R_1 \cup R_2) = A(R_1) + A(R_2)$.

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[1] Remember that on the sphere the shortest path between two points is an arc of a Great Circle. The length of a Great Circle arc can be measured in degrees; its length in linear units will depend on the radius of the circle.

For a particular geometry there are a lot of functions which satisfy properties (1)-(3). We need a "fundamental" region that can be used to build more complex regions, say, all polygons. Although squares do not work in general, **triangles** do. The process of decomposing a region into disjoint triangles is called triangulation.

III. AREA OF A SPHERICAL TRIANGLE

A. Area formula

In spherical geometry we assign the area 90 (degrees) to the spherical triangle which is an octant of the sphere. This leads to the following formula for the area of a triangle ΔABC:

$$A(\Delta ABC) = (\text{sum of the angles of } \Delta ABC) - 180.$$

The number on the right hand side is called the **spherical excess** of triangle ΔABC.

EXPLORATION 2. Verify that Spherical excess satisfies the area properties. Idea for property (3): draw a spherical triangle ΔABC, divide it into two smaller triangles (say, ΔABD and ΔADC), compute the "excess" for each of the three triangles, and verify the area property (3) in this case.

B. Area of South America

Draw a spherical triangle ΔABC to approximate the continent. The angle measures are

Angle A = ______

Angle B = ______

Angle C = ______

The Area = (spherical excess) = _______

We will convert to square units in Section V.

IV. AREA OF A SPHERICAL CIRCLE.

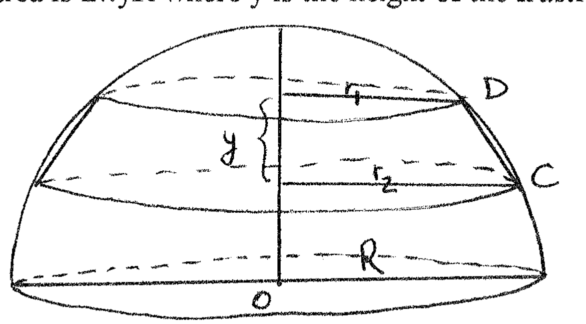
A. Area formula

A spherical circle of radius r is a "cap" of the sphere. Its area (in degrees) is

$$360(1 - \cos r).$$

B. Derivation

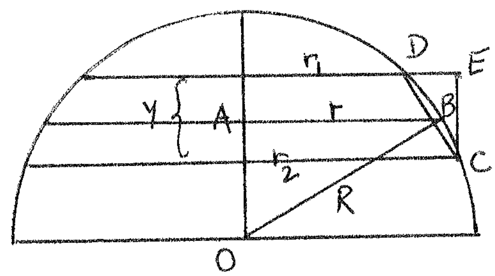
Step 1. Approximate the surface area of the "spherical' frustrum with radii r_1 and r_2, height y, cut from a sphere of radius R by the laterial area of the right circular cone of radiur R. This area is $2\pi yR$ where y is the height of the frustrum.



To derive the laterial area of the frustrum first approximate the arc CD with the chord CD (see figure above). The laterial of a frustrum of a right circular cone is

(1/2)(sum of the circumferences)(slant height).

That is, $(1/2)(2\pi r_1 + 2\pi r_2)l = 2\pi((r_1 + r_1)/2)l$.
Look at a vertical slice of the hemisphere.



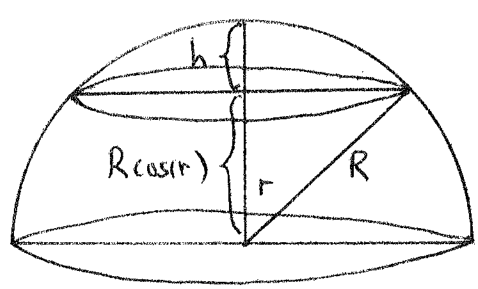
Set $r = (r_1 + r_1)/2$. Observe that $\angle$BOA is congruent to $\angle$CDE so that ΔAOB is similar to ΔEDC. Therefore,

$$\frac{r}{R} = \frac{y}{l}$$

Thus the lateral area is $2\pi((r_1 + r_1)/2)l = 2\pi rl = 2\pi yR$.

Step 2. The surface area of a spherical circle is the result of stacking together an infinitely large number of frustrums of infinitely small height. If the total "depth" of the spherical circle is h, then the area of the circle is $2\pi hR$ where R is the radiius of the sphere.

Step 3. Use trigonometry to replace the "depth" of the cap by the radius of the spherical circle: $h = R - R\cos(r) = R(1 - \cos(r))$. From Step 2, the area of the spherical circle becomes $2\pi RR(1 - \cos(r)) = 2\pi R^2(1 - \cos(r))$.



Step 4. To rewrite the formula using degrees note that the hemisphere is a circle of radius 90° so by step 3 the area of a hemisphere is $2\pi R^2$. Also, observe that a hemisphere is a union of four spherical triangles each of area 90°. Thus,

$$2\pi R^2 = 360°.$$

Replacing $2\pi R^2$ in the formula in Step 3 by 360° yields the formula: $360(1 - \cos r)$.

C. Area of Antartica

Approximate the cotinent by a spherical circle.

Radius = _____

Area (in degrees) = _______

We will convert to square units in the next section.

V. CONVERSION FROM DEGREES TO SQ UNITS

The key relationship is $2\pi R^2 = 360°$ derived in step 4 above. For the earth $R = 4000$ mi (or 6400 km). This means $1° = 276,000$ mi^2 (or 714,886 km^2).

Exercises. Using the value $1° = 276,000$ mi^2 find

A. The area of South America is ____________ mi^2.

B. The area of Antartica is ____________ mi^2.

C. If the radius of the Lenart sphere is 10 cm, find the value of 1° area on the Lenart sphere.

D. If the radius of the moon is 1750 km, find the value of 1° area on the moon.