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Boolean combinations of monadic formulas

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0. Introduction

The main concern of algebraic logic has been the development of purely algebraic proofs of results known in classical logic. The subject has not had an impact as a technique or method to solve problems of "classical logic" or to investigate properties of theories. This scarcity of applications of algebraic logic to logic was mentioned, in particular, in Monk [7].

The goal of this paper is to present a result about Boolean combinations of formulas that is stated entirely in terms of concepts of classical logic. The novel feature of this result is that it was first proved using ideas related to the theory of cylindric algebras. In particular, the method of proof illustrates how an étale space (sheaf) obtained (as in Comer [1]) by dualizing an algebra of formulas provides a natural way to view a theory.

Loosely speaking, the logical result states that a formula can be characterized syntactically (by a theory) as a Boolean combination of monadic formulas if and only if its interpretation in every model of the theory is a Boolean combination of definable subsets. More precisely,

THEOREM. *For an $\mathscr{L}$-theory Ω and a formula ψ, the following are equivalent*:

(i) *there exist a Boolean polynomial p in n variables and formulas $\phi_0, \ldots, \phi_{n-1}$, each depending on at most one free variable, such that*

$$\Omega \vdash \psi \leftrightarrow p(\phi_0, \ldots, \phi_{n-1}),$$

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(ii) *in every model* $\mathbf{M}$ *of* Ω *there exist a Boolean polynomial* p *and formulas* $\phi_0, \phi_1, \ldots$, *each depending on at most one free variable, such that*

$$\mathbf{M} \models \psi \leftrightarrow p(\phi_0, \ldots).$$

(*Note, in* (ii) *the* p*'s and* ϕ_i*'s may vary with* $\mathbf{M}$.)

The characterization provided by the theorem answers a question posed by Harry Deutsch and Dan Gallin at the Tarski Symposium in 1971; the proof, given in section 3, was obtained later that summer. After seeing the proof, especially the Boolean combinations that appeared, Gallin produced a standard model-theoretic proof. The theorem, as it is given, can be specialized and generalized in several ways. The present formulation was chosen because its proof contains all of the ingredients of the more general results but allows the reader to focus on the novel method of proof without extraneous complications.

The outline of the paper is as follows. Section 1 discusses the algebraization of logic in terms of cylindric algebras and reformulates the theorem algebraically. Section 2 shows how to view algebras of formulas as étale spaces and section 3 proves the algebraic form of the theorem. Extensions are discussed in the last section.

1. Algebraization

The first step in the proof of the theorem is to translate it into a suitable statement about cylindric algebras. For a detailed discussion of this process the reader can consult §1.1 of HMT [4] or §1 of HT [3]. Relevant facts are sketched below. The reader should be aware that "cylindric algebra" will usually mean "ω-dimensional cylindric algebra."

Suppose that $\mathscr{L}$ denotes a first-order relational language with equality symbol $\approx$ and variables $v_0, v_1, \ldots$. For logical axioms we use the system given by Tarski [8]. For the advantage of using this system see HMT [4], p. 169. The set of all formulas of $\mathscr{L}$ will be denoted by $\mathrm{Fm}(\mathscr{L})$. Each $\mathscr{L}$-theory Ω gives raise to an equivalence relation $\equiv_\Omega$ on $\mathrm{Fm}(\mathscr{L})$ by defining

$$\phi \equiv_\Omega \psi \quad \text{iff} \quad \Omega \vdash \phi \leftrightarrow \psi.$$

The $\equiv_\Omega$-equivalence class of ϕ will be denoted by $|\phi|$. The *cylindric algebra of*

$\mathscr{L}$-*formulas* associated with Ω is the system

$$\mathbf{F}_\Omega^{\mathscr{L}} = \langle \mathrm{Fm}\,(\mathscr{L})/\equiv_\Omega, +, \cdot, -, 0, 1, c_i, d_{ij}\rangle_{i,\, j<\omega}$$

where $|\phi|+|\psi| = |\phi \vee \psi|, \ldots, c_i(|\phi|) = |\exists v_i \phi|$, and $d_{ij} = |v_i \approx v_j|$. When no confusion is possible the $\mathscr{L}$ is omitted and we write $\mathbf{F}_\Omega$.

The algebra of formulas $\mathbf{F}_\Omega$ is intended to capture the syntactical properties of Ω. For semantical properties another type of cylindric algebra is needed. Given a set U the *full cylindric set algebra* (with base U) is the system

$$\mathbf{A}(U) = \langle Sb({}^\omega U), \cup, \cap, \sim, \emptyset, {}^\omega U, C_i, D_{ij}\rangle_{i,j<\omega}$$

where

$$D_{ij} = \{f \in {}^\omega U \mid f(i) = f(j)\}$$

and, for $X \subseteq {}^\omega U$,

$$C_i(X) = \{f \in {}^\omega U \mid \exists g \in X \cdot g \restriction \omega \sim \{i\} = f \restriction \omega \sim \{i\}\}.$$

For an $\mathscr{L}$-structure $\mathbf{M} = \langle M, R_0, \ldots\rangle$ and $\theta \in \mathrm{Fm}\,(\mathscr{L})$ let

$$\theta^{\mathbf{M}} = \{f \in {}^\omega M \mid \mathbf{M} \vDash \theta[f]\}.$$

The set $A_{\mathbf{M}} = \{\theta^{\mathbf{M}} \mid \theta \in \mathrm{Fm}\,(\mathscr{L})\}$ is a subuniverse of the full set algebra $\mathbf{A}(M)$ with base M; the corresponding subalgebra is called the *algebra associated with the* $\mathscr{L}$-*structure* $\mathbf{M}$ and is denoted $\mathbf{A}_{\mathbf{M}}$.

For a cylindric algebra $\mathbf{A}$ and $x \in A$, $\Delta x = \{i < \omega \mid c_i x \neq x\}$ is the *dimension set* of x. An algebra is *locally finite* (an Lf_ω) if each element has a finite dimension set. $\mathbf{F}_\Omega$ and $\mathbf{A}_{\mathbf{M}}$ are both examples of locally finite algebras. An element x in a cylindric algebra $\mathbf{A}$ is called *monadic* if $|\Delta x| \leq 1\}$. The Boolean subalgebra of $\mathbf{A}$ generated by $\{x \in A \,\big|\, |\Delta x| \leq 1\}$ is denoted $\mathbf{A}^*$. Algebras generated (as a cylindric algebra) by $\mathbf{A}^*$ have been studied by Monk [5].

The logical theorem about Boolean combinations can be restated, in the algebraic notation, as follows.

THEOREM$^{\mathrm{Alg}}$. *For an* $\mathscr{L}$-*theory* Ω *and* $\psi \in \mathrm{Fm}\,(\mathscr{L})$, *the following are equivalent*:

(i) $|\psi| \in \mathbf{F}_\Omega^*$,

(ii) for every model $\mathbf{M}$ of Ω, $\psi^{\mathbf{M}} \in \mathbf{A}_{\mathbf{M}}^*$.

2. Duality

The next goal is to describe the étale space dual of $\mathbf{F}_\Omega$. A sheaf duality theory for cylindric algebras of an arbitrary dimension, as well as for special classes of algebras such as Lf_α, was presented in Comer [1]. When specialized to algebras of formulas the construction shows that $\mathbf{F}_\Omega$ can be represented in terms of algebras $\mathbf{F}_\Sigma$ where Σ ranges over all complete and consistent extensions of Ω.[(1)] However, for a complete and consistent theory Σ, $\mathbf{F}_\Sigma$ is isomorphic to an algebra associated with a model; see HT [3], Theorem 1.12 and the remarks that follow it for a discussion of this relationship. The constructions above are composed in the lemma below to describe the syntactically defined algebra $\mathbf{F}_\Omega$ in terms of the algebras associated with its models.

For a reader unfamiliar with the notion of an étale space of structures, a brief description is included. An étale space is a pair of topological spaces X (the base space) and S (the stalk space) and a local homeomorphism $\pi : S \to X$. Figure 1 gives a picture of the setup.
When π is understood the étale space is denoted as (X, S). For each $x \in X$, $\pi^{-1}(\{x\}) = S_x$ is called the *stalk over x*.

If each stalk S_x $(x \in X)$ carries the structure of a cylindric algebra, we say that $(X, \mathbf{S})$ is an *étale space of cylindric algebras* if the (partial) operations induced on S by the operations on each stalk are continuous. Intuitively, $\mathbf{S}$ is the result of pasting together a family of cylindric algebras with disjoint universes.

A (*global*) *section* of an étale space (X, S) is a continuous function $f : X \to S$ such that πf is the identity on X. The set of all global sections of (X, S) is denoted $\Gamma(X, S)$. Note that $\Gamma(X, S) \subseteq \prod_{x \in X} S_x$. If $(X, \mathbf{S})$ is an étale space of cylindric algebras, a new algebra, denoted $\Gamma(X, \mathbf{S})$, can be formed with universe $\Gamma(X, S)$. The operations on $\Gamma(X, S)$ are defined pointwise so that $\Gamma(X, \mathbf{S})$ is a subalgebra of $\prod_{x \in X} \mathbf{S}_x$.

Now we are ready to describe the étale space $(X_\Omega, \mathbf{S}_\Omega)$ associated with a consistent theory Ω and give the connection between $\mathbf{F}_\Omega$ and $\Gamma(X_\Omega, \mathbf{S}_\Omega)$. As the base space X_Ω we take the set of all complete and consistent extensions of Ω, a Boolean space with a basis for the topology given by all sets of the form

$$\mathcal{N}_\theta = \{\Sigma \in X_\Omega \mid \Sigma \vdash \theta\}$$

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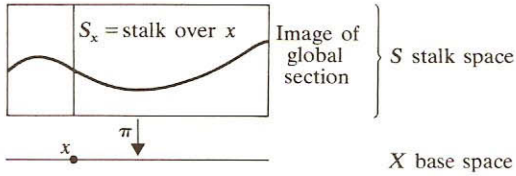
[1] Alternatively, $\mathbf{F}_\Omega$ can be viewed as a Boolean-valued model or Ω-structure (where Ω is a Boolean algebra) in the sense of Fourman and Scott [2].



where θ is an $\mathscr{L}$-sentence. To describe $\mathbf{S}_\Omega$ we first pick out the stalks. For each $\Sigma \in X_\Omega$ choose a model $\mathbf{M}(\Sigma)$ of Σ. Moreover, make the choice so that $M(\Sigma) \cap M(\Gamma) = \emptyset$ whenever $\Sigma \neq \Gamma$. Now set

$$\mathrm{S}_\Omega = \bigcup \{A_{\mathbf{M}(\Sigma)} \mid \Sigma \in X_\Omega\}$$

and define $\pi : S_\Omega \to X_\Omega$ by $\pi(a) = \Sigma$ iff $a \in A_{\mathbf{M}(\Sigma)}$. S_Ω becomes a topological space with a basis given by all sets of the form

$$\mathscr{B}_{\theta,\phi} = \{\phi^{\mathbf{M}(\Sigma)} \mid \Sigma \vdash \theta \text{ and } \Sigma \in X_\Omega\}$$

where θ is an $\mathscr{L}$-sentence and ϕ is an $\mathscr{L}$-formula.

LEMMA. (Comer [1]). *If Ω is a consistent $\mathscr{L}$-theory, then $(X_\Omega, \mathbf{S}_\Omega)$ is an étale space of locally finite cylindric algebras. Moreover,*

$$\xi : \mathbf{F}_\Omega \cong \Gamma(X_\Omega, \mathbf{S}_\Omega)$$

where the isomorphism ξ sends $|\phi|$ to $\hat{\phi}$, the global section defined for $\Sigma \in X_\Omega$ by $\hat{\phi}(\Sigma) = \phi^{\mathbf{M}(\Sigma)}$.

3. Proof of Theorem$^{\mathrm{Alg}}$

The nontrivial implication is (ii)$\Rightarrow$(i). By the Lemma $\mathbf{F}_\Omega \cong \Gamma(X_\Omega, \mathbf{S}_\Omega)$. To simplify notation we omit the subscript Ω. By (ii), for each $\Sigma \in X$ there exist a Boolean polynomial p_Σ in k_Σ variables and elements $x_{\Sigma,i} \in A_{\mathbf{M}(\Sigma)}$ with $|\Delta x_{\Sigma,i}| \leq 1$ for all $i < k_\Sigma$ such that

$$\psi^{\mathbf{M}(\Sigma)} = p_\Sigma(x_{\Sigma,0}, x_{\Sigma,1}, \ldots).$$

For each $i<k_\Sigma$ choose $f_{\Sigma,i}\in\Gamma(X,S)$ such that $f_{\Sigma,i}(\Sigma)=x_{\Sigma,i}$. Since π is a local homeomorphism there is a neighbourhood $\mathcal{W}$ of Σ in X such that, for all $\Gamma\in\mathcal{W}$

$$\psi^{\mathbf{M}(\Gamma)}=p_\Sigma(f_{\Sigma,0}(\Gamma),\ldots)$$

and

$$\Delta f_{\Sigma,i}(\Gamma)=\Delta f_{\Sigma,i}(\Sigma) \text{ contains at most one element.}$$

X is covered by such neighbourhoods, it is compact, and has a basis of clopen sets: so there is a partition $\mathcal{W}_0,\ldots,\mathcal{W}_{m-1}$ of X into clopen subsets and, for each $j<m$, $f_{j,0},\ldots,f_{j,k_j-1}\in\Gamma(\mathcal{W}_j,S)$ and Boolean polynomials $p_0,\ldots,p_{m-1}$ with the properties:

(1) for all $j<m$ and $\Sigma\in\mathcal{W}_j$, $\psi^{\mathbf{M}(\Sigma)}=p_j(f_{j,0}(\Sigma),\ldots)$;

(2) for all $j<m$ and $i<k_j$, $|\Delta f_{j,i}|\leq 1$.

For each $j<m$ choose a sentence θ_j so that $\mathcal{N}_{\theta_j}=\mathcal{W}_j$. For each $j<m$ and $i<k_j$ extends $f_{j,i}$ to a section of (X,S) and, using the isomorphism $\mathbf{F}_\Omega\cong\Gamma(X,S)$ and (2), choose formulas $\phi_{j,i}$ depending on at most one free variable such that $\hat{\phi}_{j,i}\restriction\mathcal{W}_j=f_{j,i}$. From the definitions and (1) it follows that

$$\hat{\psi}=\sum_{j<m}\hat{\theta}_j\cdot p_j(\hat{\phi}_{j,0},\hat{\phi}_{j,1},\ldots)$$

which gives (i) by the Lemma. □

Using the Lemma and the definition of $\mathbf{F}_\Omega$, the description of ψ can be translated back to $\mathcal{L}$. Namely, (ii) implies

$$\Omega\vdash\psi\ \rightarrow\bigvee_{j<m}[\theta_j\wedge p_j(\phi_{j,0},\ldots,\phi_{j,k_j-1})],$$

the right side being a Boolean combination of formulas each depending on at most one free variable.

4. Variations

Other results similar to the theorem in section 0 can be obtained using the method above.

1. (k-ary form). Essentially the same argument characterizes Boolean combinations of formulas depending upon at most k variables.

2. (Fixed Boolean combinations). When the Boolean polynomial p is specified in advance a sharper result is obtained. In this case the p_Σ's are the same for all Σ and the final description of ψ has $m = 1$ and θ_0 true (so it can be dropped).

3. (Fixed combinations and variables). Suppose p is a Boolean polynomial in n variables and $\{V_0, \ldots, V_{n-1}\}$ is a collection of nonempty sets of variables. The main argument can be modified to characterize formulas of the type $p(\phi_0, \ldots, \phi_{n-1})$ where each ϕ_i depends at most on the variables V_i.

Finally, it can be shown, by example, that it is not possible to specify, in advance, the set of variables admissible in a component (instead of a bound on the number) while allowing arbitrary Boolean combinations. The problem is that, in the models, different Boolean combinations may be used with the requisite variables, but that, globally, no single combination will work.

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